Chapter 14 H_2 optimal Control

14.1 Factorization for Simple RHP Zeros

Optimal Control v.s. Optimal Decoupling Control

Feature of optimal decoupling control: The closed-loop response is decoupled; the response of each channel is determined by only one parameter. This is important for applications:

1. When a channel is tuned it is usually desirable that other channels should not be affected
2. The feature makes the tuning of a decoupling control system significantly simplified

Feature of the optimal control in general sense: The closed-loop response may be non-decoupled. The optimal control is theoretically more important than the decoupling optimal control, since the general optimality implies that the minimum error is achieved

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Assumption

Preceding chapters: The plant was permitted to have time delays
This chapter: The plant is restricted to be **rational**
Reason: The factorization, which is the key of the H_2 optimal design, is still an open question for systems with time delays

Let $\mathbf{G}(s)$ denote an $n \times n$ plant and $\mathbf{C}(s)$ be an $n \times n$ controller. It is assumed in this chapter that

1. There is not any unstable hidden mode in $\mathbf{G}(s)$
2. $\mathbf{G}(s)$ is of full normal rank
3. $\mathbf{G}(s)$ does not have any finite zeros on the imaginary axis

The assumptions are the same as those for the H_2 decoupling control

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Inner-Outer Factorization

Inner-outer factorization: Compute the factorization of a stable nonsingular transfer function matrix $\mathbf{G}(s)$ as a product of a proper inner factor $\mathbf{G_A}(s)$ and a stable MP outer factor $\mathbf{G_{MP}}(s)$, i.e.

$$\mathbf{G}(s) = \mathbf{G_A}(s)\mathbf{G_{MP}}(s)$$

Inner matrix: A matrix $\mathbf{G_A}(s)$ is said to be inner if it is stable and satisfies $\mathbf{G_A}^*(s)\mathbf{G_A}(s) = \mathbf{I}$
$\mathbf{G_A}(s)$ is in general a full matrix. It is an **all-pass** function, since $\mathbf{G_A}^*(j\omega)\mathbf{G_A}(j\omega) = \mathbf{I}$ for all ω
Outer matrix: A matrix $\mathbf{G_{MP}}(s)$ is said to be outer if it is stable and does not have zeros in the RHP
$\mathbf{G_{MP}}(s)$ may have imaginary zeros in view of the original definition of the inner-outer factorization

Existing Problems

The inner-outer factorization is **not unique**. For example, multiplying both factors by an orthogonal matrix will not change the poles and the zeros. The obtained is still an inner-outer factorization

To obtain a unique inner-outer factorization, a constraint must be imposed on the inner factor or the outer factor. For example, let $\mathbf{G_A}(\infty) = I$

The original inner-outer factorization is **only defined for stable plants**. Aiming at the design problem of general linear systems, the extended inner-outer factorization is defined in this chapter, which is applicable to both stable and unstable plants

Extension of the Inner-Outer Factorization

Definition

$\mathbf{G_A}(s)$ and $\mathbf{G_{MP}}(s)$ are said to be the extended inner-outer factorization of $\mathbf{G}(s)$ if

1. $\mathbf{G_A}^*(s)\mathbf{G_A}(s) = \mathbf{I}$
2. $\mathbf{G_A}(s)$ and $\mathbf{G_{MP}}^{-1}(s)$ are stable
3. $\mathbf{G_{MP}}(s)$ has the same closed RHP poles as $\mathbf{G}(s)$

In the new definition, Condition 1 is from the original definition. The condition implies that $\mathbf{G_A}(s)$ has the same number of zeros and poles

Conditions 2 and 3 are from the design requirement of the H_2 optimal controller

In Condition 2 it is required that $\mathbf{G_A}(s)$ **must be stable** even if the plant is unstable, while $\mathbf{G_{MP}}(s)$ is not necessarily stable. Since $\mathbf{G_{MP}}(s)$ is required to be MP, $\mathbf{G_{MP}}^{-1}(s)$ **must be stable**

If Condition 3 is not satisfied, the plant has to be factorized a second time, so that a MP part with the same closed RHP poles as $\mathbf{G}(s)$ is obtained. Such a factorization is needed in deriving the optimal solution

In particular, to obtain the optimal controller analytically an analytical solution is expected for the factorization

The extended inner-outer factorization is a control-oriented definition. One can regard the original inner-outer factorization as a special case of the new factorization except for the imaginary axis case

In the new definition the outer factor does not have imaginary zeros anymore. Otherwise, an internally unstable system may be obtained in the design

For clarity of presentation, the factorization for plants with simple open RHP zeros is introduced first. The result will be extended to plants with multiple open RHP zeros shortly later

Assume that $z_j (j = 1, 2, ..., r_z)$ are the simple zeros of $\mathbf{G}(s)$.

Definition

The nonzero $1 \times n$ vector $\mathbf{v_j}$ satisfying $\mathbf{v_j}\mathbf{G}(z_j) = \mathbf{0}$ is called the direction of the zero z_j

In the definition, the zero is not necessarily a RHP zero

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The vector $\mathbf{v_j^T}$ is the eigenvector of $\mathbf{G}^T(z_j)$ associated with the eigenvalue zero

It is called a zero direction since for system input of the form $\mathbf{c}e^{z_j t}$ (where $\mathbf{c}$ is an arbitrary complex vector, $t \geq 0$), the output caused by the input in the direction of $\mathbf{v_j}$ is identically zero given appropriate initial conditions

Now let $\mathbf{G_A}(s)$ be an $n \times n$ transfer function matrix with the following features:

1. $\det[\mathbf{G_A}(s)]$ is not identically zero (or equivalently, $\mathbf{G_A}(s)$ is of full normal rank)
2. $\det[\mathbf{G_A}(s)]$ does not have any poles at z_j

Lemma

$\mathbf{G_A}(s)$ has a simple zero z_j with zero direction $\mathbf{v_j}$ if and only if $\mathbf{G_A}^{-1}(s)$ has an expression of the form

$$\mathbf{G_A}^{-1}(s) = (-s + z_j)^{-1}\alpha_{\mathbf{j}}\mathbf{v_j} + \mathbf{G_0}(s)$$

where $\mathbf{G_0}(s)$ is a term without poles at z_j and $\alpha_{\mathbf{j}}$ is a nonzero $n \times 1$ vector

Proof.

Suppose that $\mathbf{G_A}^{-1}(s)$ has an expression of the form

$$\mathbf{G_A}^{-1}(s) = (-s + z_j)^{-1}\alpha_{\mathbf{j}}\mathbf{v_j} + \mathbf{G_0}(s)$$

where $\mathbf{G_0}(s)$ is a term without poles at z_j

Proof ctd.1.

Multiply both sides of the equation on the right by $(-s+z_j)\mathbf{G_A}(s)$. We have

$$(-s+z_j)\mathbf{I} = [\alpha_\mathbf{j}\mathbf{v_j} + (-s+z_j)\mathbf{G_0}(s)]\mathbf{G_A}(s)$$

As $\alpha_\mathbf{j}$ is a nonzero column vector, one obtains that $\mathbf{v_j}\mathbf{G_A}(z_j) = \mathbf{0}$. This implies that $\mathbf{G_A}(s)$ has at least one zero at z_j.

On the other hand, taking determinants yields

$$(-s+z_j)^n = \det[\alpha_\mathbf{j}\mathbf{v_j} + (-s+z_j)\mathbf{G_0}(s)]\det[\mathbf{G_A}(s)]$$

Since the rows of $\alpha_\mathbf{j}\mathbf{v_j}$ are multiples of $\mathbf{v_j}$, $\alpha_\mathbf{j}\mathbf{v_j}$ is of rank 1. It is known that $\det[\alpha_\mathbf{j}\mathbf{v_j} + (-s+z_j)\mathbf{G_0}(s)]$ has a zero at z_j of multiplicity at least $n-1$. It is deduced that $\det[\mathbf{G_A}(s)]$ has a zero at z_j of multiplicity at most 1

Proof ctd.2.

Therefore, z_j is a simple zero of $\mathbf{G_A}(s)$

Conversely, suppose that $\mathbf{G_A}(s)$ has a simple zero z_j with the zero direction $\mathbf{v_j}$. Since $\det[\mathbf{G_A}(s)]$ is not identically zero, $\mathbf{G_A}(s)$ is invertible and ${\mathbf{G_A}}^{-1}(s)$ has a simple pole at z_j. Apply the Laurent expression to rational transfer function matrices. ${\mathbf{G_A}}^{-1}(s)$ can be written in the following form:

$$\begin{aligned} {\mathbf{G_A}}^{-1}(s) &= \sum_{i=-1}^{\infty} (-s+z_j)^i \mathbf{R_{i+2}} \\ &= (-s+z_j)^{-1} \sum_{i=-1}^{\infty} (-s+z_j)^{i+1} \mathbf{R_{i+2}} \end{aligned}$$

Proof ctd.3.

$\mathbf{R_i}(i = 1, 2, ..., \infty)$ are $n \times n$ constant matrices. As $\mathbf{G_A}(s)$ does not have poles at z_j, $\mathbf{R_1} \neq \mathbf{0}$

Choose a $1 \times n$ vector β satisfying $\beta\mathbf{R_1} \neq \mathbf{0}$. We have

$$\beta\mathbf{G_A}^{-1}(s) = (-s + z_j)^{-1} \sum_{i=-1}^{\infty} \beta(-s + z_j)^{i+1}\mathbf{R_{i+2}}$$

Rewrite the equation as

$$(-s + z_j)\beta = \left[\sum_{i=-1}^{\infty} \beta(-s + z_j)^{i+1}\mathbf{R_{i+2}}\right] \mathbf{G_A}(s)$$

Proof ctd.4.

In particular, at $s = z_j$ we have

$$\beta \mathbf{R_1} \mathbf{G_A}(z_j) = \mathbf{0}$$

Recall that the definition of the zero direction is

$$\mathbf{v_j} \mathbf{G}(z_j) = \mathbf{0}.$$

It is easy to see that $\beta \mathbf{R_1}$ is a multiple of $\mathbf{v_j}$ for any row vector β. This happens if and only if there exists a nonzero column vector $\boldsymbol{\alpha}_\mathbf{j}$ so that $\mathbf{R_1} = \alpha_\mathbf{j} \mathbf{v_j}$ □

In the theorem, $\alpha_\mathbf{j} \mathbf{v_j}$ is in fact the residue of $\mathbf{G_A}^{-1}(s)$ at z_j. The uniqueness of the residue implies that the expression is unique

Proof ctd.4.

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In the theorem, $\alpha_\mathbf{j} \mathbf{v_j}$ is in fact the residue of ${\mathbf{G_A}}^{-1}(s)$ at z_j. The uniqueness of the residue implies that the expression is unique

Lemma

If an $n \times n$ stable transfer function matrix $\mathbf{G_A}(s)$ satisfies

1. $\mathbf{G_A}^*(s)\mathbf{G_A}(s) = \mathbf{I}$, *and*
2. $z_j(Re(z_j) > 0, j = 1, 2, ..., r_z)$ *are the only open RHP zeros of* $\mathbf{G_A}(s)$ *with zero directions* v_j,

then $\mathbf{G_A}(s)$ *is the inner factor of* $\mathbf{G}(s)$ *and*
$\mathbf{G_{MP}}(s) = \mathbf{G_A}^{-1}(s)\mathbf{G}(s)$ *is the outer factor*

Proof.

It is enough to prove that $\mathbf{G_{MP}}(s)$ is MP and has the same closed RHP poles as $\mathbf{G}(s)$

Since $z_j(j = 1, 2, ..., r_z)$ are the only open RHP zeros of $\mathbf{G_A}(s)$, the only open RHP poles of $\mathbf{G_A}^{-1}(s)$ are z_j

Proof ctd.1.

According to Lemma 3, $\mathbf{G_A}^{-1}(s)$ has an expression of the form

$$\mathbf{G_A}^{-1}(s) = (-s + z_j)^{-1}\alpha_\mathbf{j}\mathbf{v_j} + \mathbf{G_0}(s)$$

where $\alpha_\mathbf{j}$ is a nonzero column vector and $\mathbf{G_0}(s)$ is a term without poles at z_j.
Multiply both sides on the right by $\mathbf{G}(s)$. We have

$$\mathbf{G_A}^{-1}(s)\mathbf{G}(s) = (-s + z_j)^{-1}\alpha_\mathbf{j}\mathbf{v_j}\mathbf{G}(s) + \mathbf{G_0}(s)\mathbf{G}(s)$$

The second term in the right-hand side is analytic at z_j for $j = 1, 2, ..., r_z$, since both $\mathbf{G_0}(s)$ and $\mathbf{G}(s)$ do not have poles at z_j

Proof ctd.2.

In the first term, $\mathbf{v_j}\mathbf{G}(z_j) = \mathbf{0}$. Hence, ${\mathbf{G_A}}^{-1}(s)\mathbf{G}(s)$ does not have a pole at z_j for $j = 1, 2, ..., r_z$; that is, all open RHP poles of ${\mathbf{G_A}}^{-1}(s)$ are cancelled by those open RHP zeros of $\mathbf{G}(s)$. $\mathbf{G_{MP}}(s) = {\mathbf{G_A}}^{-1}(s)\mathbf{G}(s)$ has the same closed RHP poles as $\mathbf{G}(s)$

Furthermore, those open RHP zeros of $\mathbf{G}(s)$ are the only possible open RHP zeros of $\mathbf{G_{MP}}(s)$. All of these zeros are cancelled. Therefore, $\mathbf{G_{MP}}(s)$ is MP

$\square$

The following theorem gives the main result of this section

Theorem

The $n \times n$ transfer function matrix $\mathbf{G_A}(s)$ of the form

$$\mathbf{G_A}(s) = \mathbf{I} - \mathbf{B}^*(s\mathbf{I} + \bar{\mathbf{A}})^{-1}\mathbf{F}^{-1}\mathbf{B}$$

is inner. Here

$$\mathbf{A} = \begin{bmatrix} z_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & z_{r_z} \end{bmatrix}, \mathbf{B} = \begin{bmatrix} \mathbf{v_1} \\ \vdots \\ \mathbf{v_{r_z}} \end{bmatrix},$$

$$\mathbf{F} = [f_{ij}]\,, f_{ij} = \frac{\mathbf{v_i}\mathbf{v_j}^*}{\bar{z}_j + z_i}, i, j = 1, 2, ..., r_z$$

Proof.

First, it is shown that $\mathbf{G_A}^*(s)\mathbf{G_A}(s) = \mathbf{I}$. $z_j(j = 1, 2, ..., r_z)$ are the simple zeros of $\mathbf{G_A}(s)$. By using this fact, it can be proved that

$$\mathbf{G_A}^*(s) = \mathbf{I} - \mathbf{B}^*(\mathbf{F}^*)^{-1}(-s\mathbf{I} + \mathbf{A}^T)^{-1}\mathbf{B}$$

Since $\mathbf{F} = \mathbf{F}^*$,

$$\begin{aligned} & \mathbf{G_A}^*(s)\mathbf{G_A}(s) \\ = \; & [\mathbf{I} - \mathbf{B}^*\mathbf{F}^{-1}(-s\mathbf{I} + \mathbf{A}^T)^{-1}\mathbf{B}][\mathbf{I} - \mathbf{B}^*(s\mathbf{I} + \bar{\mathbf{A}})^{-1}\mathbf{F}^{-1}\mathbf{B}] \end{aligned}$$

Proof ctd.1.

$$
\begin{aligned}
= \;& \mathbf{I} - \mathbf{B}^*\mathbf{F}^{-1}(-s\mathbf{I} + \mathbf{A}^T)^{-1}\mathbf{B} - \mathbf{B}^*(s\mathbf{I} + \bar{\mathbf{A}})^{-1}\mathbf{F}^{-1}\mathbf{B} + \\
& \mathbf{B}^*\mathbf{F}^{-1}(-s\mathbf{I} + \mathbf{A}^T)^{-1}\mathbf{B}\mathbf{B}^*(s\mathbf{I} + \bar{\mathbf{A}})^{-1}\mathbf{F}^{-1}\mathbf{B} \\
= \;& \mathbf{I} - \mathbf{B}^*\mathbf{F}^{-1}(-s\mathbf{I} + \mathbf{A}^T)^{-1}\{\mathbf{F}(s\mathbf{I} + \bar{\mathbf{A}}) + \\
& (-s\mathbf{I} + \mathbf{A}^T)\mathbf{F} - \mathbf{B}\mathbf{B}^*\}(s\mathbf{I} + \bar{\mathbf{A}})^{-1}\mathbf{F}^{-1}\mathbf{B} \\
= \;& \mathbf{I} - \mathbf{B}^*\mathbf{F}^{-1}(-s\mathbf{I} + \mathbf{A}^T)^{-1}(\mathbf{F}\bar{\mathbf{A}} + \mathbf{A}^T\mathbf{F} - \mathbf{B}\mathbf{B}^*)(s\mathbf{I} + \bar{\mathbf{A}})^{-1}\mathbf{F}^{-1}\mathbf{B}
\end{aligned}
$$

It can be verified that $\mathbf{F}$ satisfies the following Lyapunov equation:

$$\mathbf{F}\bar{\mathbf{A}} + \mathbf{A}^T\mathbf{F} = \mathbf{B}\mathbf{B}^*$$

One readily obtains $\mathbf{G_A}^*(s)\mathbf{G_A}(s) = \mathbf{I}$

Proof ctd.2.

Next, examine the zeros of $\mathbf{G_A}(s)$. Since

$$\mathbf{G_A}^{-1}(s) = \mathbf{G_A}^{*}(s) = \mathbf{I} - \mathbf{B}^{*}\mathbf{F}^{-1}(-s\mathbf{I} + \mathbf{A}^{T})^{-1}\mathbf{B}$$

$z_j (j = 1, 2, ..., r_z)$ are the only open RHP zeros of $\mathbf{G_A}(s)$. By Lemma 4 it is known that $\mathbf{G_A}(s)$ is inner $\square$

Evidently, $\mathbf{G_A}(s)$ is unique since $\mathbf{G_A}(\infty) = \mathbf{I}$

14.2 Construction Procedure of Factorizations

One may wish to know how the inner matrix in the last section was conceived. The procedure will be explained in detail in this section

An important step in constructing: Construct a transfer function matrix with the desired zero-pole distribution and, at the same time, without the RHP zero-pole cancellation

Difficulty: This is very easy in scalar systems; in multivariable systems the constructing procedure is an involved problem, since the zero of the plant may not be the zero of its elements

In the last chapter, it was shown that the inner matrix could be obtained by analyzing the inverse of a transfer function matrix, since all zeros of the plant would emerge in the inverse matrix

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This is an important idea that inspires the solving of the factorization problem in this chapter

It is found in interpolation study that the zero-pole distribution of a transfer function matrix is closely related to the partial fraction expansion of its inverse matrix

This fact will be used to compute the factorization

To obtain a unique inner-outer factorization, let $\mathbf{G_A}(\infty) = I$. Assume that $z_j (j = 1, 2, ..., r_z)$ are the simple zeros of $\mathbf{G_A}(s)$ with the zero directions $\mathbf{v_j}$. By Liouville's Theorem, $\mathbf{G_A}^{-1}(s)$ has the following partial fraction expansion:

$$\mathbf{G_A}^{-1}(s) = \mathbf{I} + \sum_{j=1}^{r_z} (-s + z_j)^{-1} \alpha_\mathbf{j} \mathbf{v_j}$$

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$$
{\mathbf{G_A}}^{-1}(s) = \mathbf{I} + \sum_{j=1}^{r_z} (-s + z_j)^{-1} \alpha_\mathbf{j} \mathbf{v_j}
$$

where α_j are unknown nonzero $n \times 1$ vectors. Based on this expression, the form of $\mathbf{G_A}(s)$ is given in the following lemma

Lemma

If $\mathbf{G_A}^{-1}(s) = \mathbf{I} + \sum_{j=1}^{r_z} (-s + z_j)^{-1} \boldsymbol{\alpha}_\mathbf{j} \mathbf{v_j}$, *it can be proved that*

$$\mathbf{G_A}(s) = \mathbf{I} + \mathbf{C_v}[s\mathbf{I} - (\mathbf{A_v} + \mathbf{B_v}\mathbf{C_v})]^{-1}\mathbf{B_v}$$

where

$$\mathbf{A_v} = \begin{bmatrix} z_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & z_{r_z} \end{bmatrix},$$

$$\mathbf{B_v} = [\mathbf{v_1}^T, \mathbf{v_2}^T, \cdots, \mathbf{v_{r_z}}^T]^T, \mathbf{C_v} = [\boldsymbol{\alpha}_\mathbf{1}, \boldsymbol{\alpha}_\mathbf{2}, \cdots, \boldsymbol{\alpha}_{\mathbf{r_z}}]$$

Proof.

$$
\begin{aligned}
& \left[\mathbf{I}-\sum_{j=1}^{r_z}(s-z_j)^{-1}\alpha_{\mathbf{j}}\mathbf{v_j}\right]\left[\mathbf{I}+\mathbf{C_v}[s\mathbf{I}-(\mathbf{A_v}+\mathbf{B_v}\mathbf{C_v})]^{-1}\mathbf{B_v}\right] \\
= \;& \mathbf{I}-\sum_{j=1}^{r_z}(s-z_j)^{-1}\alpha_{\mathbf{j}}\mathbf{v_j}+\mathbf{C_v}[s\mathbf{I}-(\mathbf{A_v}+\mathbf{B_v}\mathbf{C_v})]^{-1}\mathbf{B_v}- \\
& \left[\sum_{j=1}^{r_z}(s-z_j)^{-1}\alpha_{\mathbf{j}}\mathbf{v_j}\right]\mathbf{C_v}[s\mathbf{I}-(\mathbf{A_v}+\mathbf{B_v}\mathbf{C_v})]^{-1}\mathbf{B_v} \\
= \;& \mathbf{I}-\mathbf{C_v}(s\mathbf{I}-\mathbf{A_v})^{-1}\mathbf{B_v}+\mathbf{C_v}[s\mathbf{I}-(\mathbf{A_v}+\mathbf{B_v}\mathbf{C_v})]^{-1}\mathbf{B_v}- \\
& \mathbf{C_v}(s\mathbf{I}-\mathbf{A_v})^{-1}\mathbf{B_v}\mathbf{C_v}[s\mathbf{I}-(\mathbf{A_v}+\mathbf{B_v}\mathbf{C_v})]^{-1}\mathbf{B_v} \\
= \;& \mathbf{I}-\mathbf{C_v}(s\mathbf{I}-\mathbf{A_v})^{-1}\left\{[s\mathbf{I}-(\mathbf{A_v}+\mathbf{B_v}\mathbf{C_v})]-(s\mathbf{I}-\mathbf{A_v})+\mathbf{B_v}\mathbf{C_v}\right\} \\
& [s\mathbf{I}-(\mathbf{A_v}+\mathbf{B_v}\mathbf{C_v})]^{-1}\mathbf{B_v} \\
= \;& \mathbf{I}
\end{aligned}
$$

Since α_j are unknown, more conditions are needed to compute $\mathbf{G_A}(s)$

Consider the transfer function matrix with the same number of zeros and poles, since the inner factor possesses the feature. Assume that $p_j(j = 1, 2, ..., r_z)$ are simple zeros of $\mathbf{G_A}^{-1}(s)$ with the zero directions $\mathbf{w_j}$, and $z_j \neq p_j$. Then p_j are simple poles of $\mathbf{G_A}(s)$.

On one hand, from Lemma 3 and Liouville's Theorem $\mathbf{G_A}(s)$ can be expressed as

$$\mathbf{G_A}(s) = \mathbf{I} + \sum_{j=1}^{r_z} (s - p_j)^{-1} \mathbf{w_j} \boldsymbol{\beta}_\mathbf{j}$$

where β_j are unknown nonzero $1 \times n$ vectors

The expression can be rewritten as

$$\mathbf{G_A}(s) = \mathbf{I} + \mathbf{C_w}(s\mathbf{I} - \mathbf{A_w})^{-1}\mathbf{B_w}$$

where

$$\mathbf{A_w} = \begin{bmatrix} p_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & p_{r_z} \end{bmatrix},$$

$$\mathbf{B_w} = [\boldsymbol{\beta_1}^T, \boldsymbol{\beta_2}^T, \cdots, \boldsymbol{\beta_{r_z}}^T]^T, \mathbf{C_w} = [\mathbf{w_1}, \mathbf{w_2}, \cdots, \mathbf{w_{r_z}}]$$

On the other hand, the poles of $\mathbf{G_A}(s)$ are the eigenvalues of the matrix $\mathbf{A_v} + \mathbf{B_v}\mathbf{C_v}$. In other words, $p_j(j = 1, 2, ..., r_z)$ are the eigenvalues of $\mathbf{A_v} + \mathbf{B_v}\mathbf{C_v}$. This implies that $\mathbf{A_v} + \mathbf{B_v}\mathbf{C_v}$ is similar to $\mathbf{A_w}$, or equivalently, there exists an invertible matrix $\mathbf{F}$ such that

$$\mathbf{A_v} + \mathbf{B_v}\mathbf{C_v} = \mathbf{F}\mathbf{A_w}\mathbf{F}^{-1}$$

Substitute this expression into $\mathbf{G_A}(s)$. One obtains that

$$\begin{aligned}
\mathbf{G_A}(s) &= \mathbf{I} + \mathbf{C_v}[s\mathbf{I} - (\mathbf{A_v} + \mathbf{B_v}\mathbf{C_v})]^{-1}\mathbf{B_v} \\
&= \mathbf{I} + \mathbf{C_v}(s\mathbf{I} - \mathbf{F}\mathbf{A_w}\mathbf{F}^{-1})^{-1}\mathbf{B_v} \\
&= \mathbf{I} + \mathbf{C_v}\mathbf{F}(s\mathbf{I} - \mathbf{A_w})^{-1}\mathbf{F}^{-1}\mathbf{B_v}
\end{aligned}$$

The two expressions are identical when $\mathbf{C_w} = \mathbf{C_v}\mathbf{F}$ and $\mathbf{B_w} = \mathbf{F}^{-1}\mathbf{B_v}$. If $\mathbf{F}$ is known, $\mathbf{G_A}(s)$ can readily be derived (Figure)

Consider the computation of $\mathbf{F}$. Substituting the expressions of $\mathbf{C_w}$ into (1) yields the following Lyapunov equation:

$$\mathbf{F}\mathbf{A_w} - \mathbf{A_v}\mathbf{F} = \mathbf{B_v}\mathbf{C_w}$$

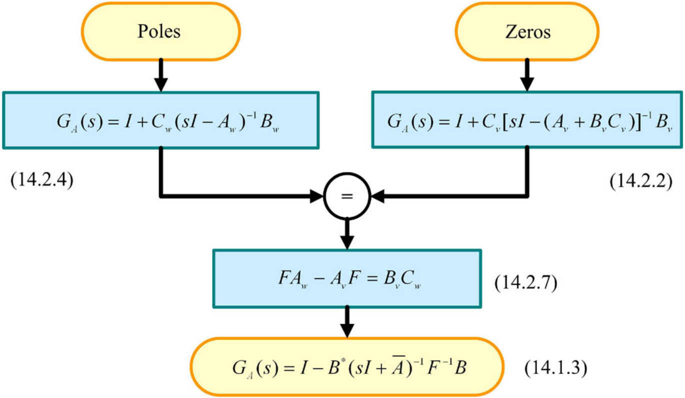

Figure: Construction of $\mathbf{G_A}(s)$

Let $f_{ij}(i,j=1,2,...,r_z)$ be the (i,j)th element of $\mathbf{F}$. Evidently, the matrix equation is equivalent to the following scalar equations:

$$f_{ij}p_j - z_i f_{ij} = \mathbf{v_i w_j}$$

The unique solution of this equation is

$$f_{ij} = \frac{\mathbf{v_i w_j}}{p_j - z_i}$$

The matrix $\mathbf{F}$ is now obtained.
When $\mathbf{B_v}$ and $\mathbf{F}$ are known, $\mathbf{B_w} = \mathbf{F}^{-1}\mathbf{B_v}$, or equivalently,

$$\begin{bmatrix} \beta_{\mathbf{1}} \\ \vdots \\ \beta_{\mathbf{r_z}} \end{bmatrix} = \begin{bmatrix} f_{11} & \cdots & f_{1r_z} \\ \vdots & \ddots & \vdots \\ f_{r_z1} & \cdots & f_{r_zr_z} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{v_1} \\ \vdots \\ \mathbf{v_{r_z}} \end{bmatrix}$$

The computing procedure is summarized as the following theorem

Theorem

Assume that $z_j (j = 1, 2, ..., r_z)$ *are the simple zeros of* $\mathbf{G_A}(s)$ *with the zero directions* $\mathbf{v_j}$, p_j *are simple zeros of* $\mathbf{G_A}^{-1}(s)$ *with the zero directions* $\mathbf{w_j}$, $z_j \neq p_j$, *and* $\mathbf{G_A}(\infty) = \mathbf{I}$. *Then* $\mathbf{G_A}(s)$ *can be written as*

$$\mathbf{G_A}(s) = \mathbf{I} + \sum_{j=1}^{r_z} (s - p_j)^{-1} \mathbf{w_j} \beta_\mathbf{j}$$

where

$$\begin{bmatrix} \beta_\mathbf{1} \\ \vdots \\ \beta_\mathbf{r_z} \end{bmatrix} = \mathbf{F}^{-1} \begin{bmatrix} \mathbf{v_1} \\ \vdots \\ \mathbf{v_{r_z}} \end{bmatrix},$$

$$\mathbf{F} = [f_{ij}], f_{ij} = \frac{\mathbf{v}_i \mathbf{w}_j}{p_j - z_i}, i, j = 1, 2, ..., r_z$$

Proof.

Follows the foregoing analysis procedure. □

Consider a special case: $z_j(\mathrm{Re}(z_j) > 0, j = 1, 2, ..., r_z)$ are the simple zeros, and the only open RHP zeros, of $\mathbf{G_A}(s)$ with zero directions $\mathbf{v_j}$, while $p_j = -\bar{z}_j$ are the simple zeros of $\mathbf{G_A}^{-1}(s)$ with zero directions $\mathbf{w_j} = -\mathbf{v_j}$

This is the case encountered in the extended inner-outer factorization problem.

Lemma

Assume that $\mathbf{G_A}(s)$ is the inner factor of $\mathbf{G}(s)$. If z_j are the simple open RHP zeros of $\mathbf{G_A}(s)$, then $-\bar{z}_j$ are the simple poles of $\mathbf{G_A}(s)$

Proof.

Follows the foregoing analysis procedure. □

Consider a special case: $z_j(\mathrm{Re}(z_j) > 0, j = 1, 2, ..., r_z)$ are the simple zeros, and the only open RHP zeros, of $\mathbf{G_A}(s)$ with zero directions $\mathbf{v_j}$, while $p_j = -\bar{z}_j$ are the simple zeros of $\mathbf{G_A}^{-1}(s)$ with zero directions $\mathbf{w_j} = -\mathbf{v_j}$

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Proof.

$$\begin{aligned}\det[\mathbf{G_A}^*(s)] &= \det[\mathbf{G_A}^T(-s)] \\ &= \det[\mathbf{G_A}(-s)]\end{aligned}$$

It is easy to know that $-z_j$ are the simple zeros of $\mathbf{G_A}^*(s)$. The coefficients in the zero polynomial of $\mathbf{G_A}^*(s)$ are real. This implies that $-\bar{z}_j$ are the simple zeros of $\mathbf{G_A}^*(s)$, too

Since $\mathbf{G_A}^*(s)\mathbf{G_A}(s) = \mathbf{I}$,

$$\mathbf{G_A}^{-1}(s) = \mathbf{G_A}^*(s)$$

$-\bar{z}_j$ are simple zeros of $\mathbf{G_A}^{-1}(s)$ and thus simple poles of $\mathbf{G_A}(s)$ □

Theorem

Assume that $z_j (j = 1, 2, ..., r_z)$ *are the simple zeros of* $\mathbf{G_A}(s)$ *with the zero directions* $\mathbf{v_j}$, $-\bar{z}_j$ *are simple zeros of* $\mathbf{G_A}^{-1}(s)$ *with the zero directions* $-\mathbf{v_j}$, *and* $\mathbf{G_A}(\infty) = \mathbf{I}$. *Then the following transfer function matrix is inner:*

$$\mathbf{G_A}(s) = \mathbf{I} + \sum_{j=1}^{r_z} (s + \bar{z}_j)^{-1} \mathbf{v_j}^* \beta_\mathbf{j}$$

where

$$\begin{bmatrix} \beta_\mathbf{1} \\ \vdots \\ \beta_{\mathbf{r_z}} \end{bmatrix} = \mathbf{F}^{-1} \begin{bmatrix} \mathbf{v_1} \\ \vdots \\ \mathbf{v_{r_z}} \end{bmatrix}$$

$$\mathbf{F} = [f_{ij}], f_{ij} = \frac{\mathbf{v_i v_j}^*}{\bar{z}_j + z_i}, i, j = 1, 2, ..., r_z$$

Let

$$\mathbf{A} = \begin{bmatrix} z_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & z_{r_z} \end{bmatrix}, \mathbf{B} = \begin{bmatrix} \mathbf{v_1} \\ \vdots \\ \mathbf{v_{r_z}} \end{bmatrix}$$

$\mathbf{G_A}(s)$ can be expressed in the matrix form in the last section

$\mathbf{G_A}^*(s) = \mathbf{G_A}^{-1}(s)$. By the first Lemma in this section, $\mathbf{G_A}^*(s)$ can be expressed in the matrix form in the last section

14.3 Factorizations for Multiple RHP Zeros

In this section, the factorization for simple zeros will be extended to the case of multiple zeros. More precisely, the parameters **A**, **B**, and **F** will be derived for plants with multiple RHP zeros

Observation: In the factorization of plants with simple RHP zeros, it is seen that the construction of **A** depends on the open RHP zeros, while the construction of **B** and the computation of **F** depend on zero directions

Problems: The zero direction is only defined for simple zeros
Solution: Define the zero direction for multiple zeros and then construct **A** and **B**, and compute **F** based on the definition

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Solution: Define the zero direction for multiple zeros and then construct **A** and **B**, and compute **F** based on the definition

Definition

Assume that z_j is a k_j multiplicity zero of $\mathbf{G}(s)$. The nonzero $1 \times n$ vectors $\mathbf{v_{jk}}(k = 1, 2, ..., k_j)$ satisfying

$$\lim_{s \to z_j} \frac{d^l}{ds^l} \left\{ \left[\sum_{i=-k_j}^{-1} \mathbf{v_{j(i+k_j+1)}} (-s + z_j)^{i+k_j} \right] \mathbf{G}(s) \right\} = 0,$$
$$l = 0, 1, ..., k_j - 1$$

are called the zero directions of z_j

The definition of multiple zero directions is a natural extension of the original definition of the zero direction. When $k_j = 1$, $\mathbf{v}_{j1}\mathbf{G}(z_j) = \mathbf{0}$. This definition reduces to the one for simple zeros

Assume that $z_j (j = 1, 2, ..., r_z)$ are k_j multiplicity RHP zeros of $\mathbf{G}(s)$ with zero directions $\mathbf{v_{jk}} (k = 1, 2, ..., k_j)$

A special case is that some z_j are the common zeros of all elements in $\mathbf{G}(s)$. In this case, the common zero in $\mathbf{G}(s)$ should be separated before the extended inner-outer factorization is carried out. Otherwise, $\mathbf{G}(z_j) = \mathbf{0}$; the corresponding zero direction can be any nonzero vector

This can be achieved by removing the following factor from $\mathbf{G}(s)$:

$$\frac{-s/\bar{z}_j + 1}{s/z_j + 1}$$

and then factorize the remainder of $\mathbf{G}(s)$:

$$\frac{s/z_j + 1}{-s/\bar{z}_j + 1}\mathbf{G}(s) = \mathbf{G_A}(s)\mathbf{G_{MP}}(s)$$

The inner factor of the original plant $\mathbf{G}(s)$ is

$$\frac{-s/\bar{z}_j+1}{s/z_j+1}\mathbf{G}_{\mathbf{A}}(s)$$

It should be emphasized that only those common zeros are separated in the procedure. Some z_j are the zeros of $\mathbf{G}(s)$ at the same place, rather than the common zeros of all elements in

$$\frac{s/z_j+1}{-s/\bar{z}_j+1}\mathbf{G}(s)$$

These zeros should be preserved in $\mathbf{G}(s)$

To simplify the presentation, it is assumed that $\mathbf{G}(z_j) \neq \mathbf{0}$

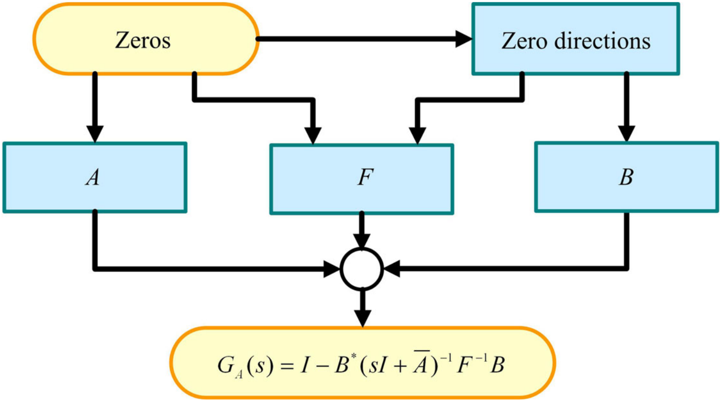

Figure: Computation of the inner factor

Let $\mathbf{G_A}(s)$ be an $n \times n$ transfer function matrix. $\det[\mathbf{G_A}(s)]$ is not identically zero and $\mathbf{G_A}(s)$ does not have poles at z_j

Lemma

$\mathbf{G_A}(s)$ has a k_j multiplicity zeros z_j with zero directions $\mathbf{v_{jk}}(k = 1, 2, \ldots, k_j)$ if and only if ${\mathbf{G_A}}^{-1}(s)$ can be expressed as

$$\begin{aligned} {\mathbf{G_A}}^{-1}(s) &= (-s + z_j)^{-k_j}\boldsymbol{\alpha}_\mathbf{j}\mathbf{v_{j1}} + \ldots + \\ &\quad (-s + z_j)^{-2}\boldsymbol{\alpha}_\mathbf{j}\mathbf{v_{j(k_j-1)}} + \\ &\quad (-s + z_j)^{-1}\boldsymbol{\alpha}_\mathbf{j}\mathbf{v_{jk_j}} + \mathbf{G_0}(s) \end{aligned}$$

where $\boldsymbol{\alpha}_\mathbf{j}$ is a nonzero column vector and $\mathbf{G_0}(s)$ is a term without poles at z_j

Proof.

Suppose first that $\mathbf{G_A}^{-1}(s)$ can be expressed as

$$\begin{aligned}\mathbf{G_A}^{-1}(s) &= (-s+z_j)^{-k_j}\boldsymbol{\alpha}_\mathbf{j}\mathbf{v_{j1}} + \ldots + \\ &\quad (-s+z_j)^{-2}\boldsymbol{\alpha}_\mathbf{j}\mathbf{v_{j(k_j-1)}} + \\ &\quad (-s+z_j)^{-1}\boldsymbol{\alpha}_\mathbf{j}\mathbf{v_{jk_j}} + \mathbf{G_0}(s)\end{aligned}$$

Multiply both sides of the equation on the right by $(-s+z_j)^{k_j}\mathbf{G_A}(s)$. We have

$$\begin{aligned}(-s+z_j)^{k_j}\mathbf{I} &= [\boldsymbol{\alpha}_\mathbf{j}\mathbf{v_{j1}} + \ldots + (-s+z_j)^{k_j-2}\boldsymbol{\alpha}_\mathbf{j}\mathbf{v_{j(k_j-1)}} + \\ &\quad (-s+z_j)^{k_j-1}\boldsymbol{\alpha}_\mathbf{j}\mathbf{v_{jk_j}} + (-s+z_j)^{k_j}\mathbf{G_0}(s)]\mathbf{G_A}(s)\end{aligned}$$

Compute the lth ($l = 0, 1, \ldots, k_j - 1$) derivatives of the two sides at z_j

Proof ctd.1.

$l = 0$:

$$\mathbf{v_{j1}} \mathbf{G_A}(z_j) = \mathbf{0}$$

which implies that $\mathbf{G_A}(s)$ loses rank at z_j

$l = 1$:

$$\mathbf{v_{j2}} \mathbf{G_A}(z_j) + (-1)\mathbf{v_{j1}} \frac{d}{ds} \mathbf{G_A}(z_j) = \mathbf{0}$$

which implies that $\frac{d}{ds}\mathbf{G_A}(s)$ loses rank at z_j

...

$l = k_j - 1$:

$$\mathbf{v_{jk_j}} \mathbf{G_A}(z_j) + \sum_{i=1}^{k_j-1} (-1)^{k_j-i} \frac{\mathbf{v_{ji}}}{(k_j-i)!} \frac{d^{k_j-i}}{ds^{k_j-i}} \mathbf{G_A}(z_j) = 0$$

Proof ctd.2.

which implies that $\frac{d^{k_j-i}}{ds^{k_j-i}}\mathbf{G_A}(s)$ loses rank at z_j. Hence, $\mathbf{G_A}(s)$ has at least k_j zeros at z_j

On the other hand, taking determinants of both sides yields

$$\begin{aligned}(-s+z_j)^{k_j\times n} \quad = \quad & \det[\alpha_\mathbf{j}\mathbf{v_{j1}} + \ldots + (-s+z_j)^{k_j-2}\alpha_\mathbf{j}\mathbf{v_{j(k_j-1)}} + \\ & (-s+z_j)^{k_j-1}\alpha_\mathbf{j}\mathbf{v_{jk_j}} + (-s+z_j)^{k_j}\mathbf{G_0}(s)]\det[\mathbf{G_A}(s)]\end{aligned}$$

Since all of the lth $(l = 0, 1, \ldots, k_j - 1)$ derivatives of

$$\begin{aligned}&\alpha_\mathbf{j}\mathbf{v_{j1}} + \ldots + (-s+z_j)^{k_j-2}\alpha_\mathbf{j}\mathbf{v_{j(k_j-1)}} + \\ &(-s+z_j)^{k_j-1}\alpha_\mathbf{j}\mathbf{v_{jk_j}} + (-s+z_j)^{k_j}\mathbf{G_0}(s)\end{aligned}$$

Proof ctd.3.

at z_j are of rank 1, by Theorem in Section 10.1

$$\det[\alpha_\mathbf{j}\mathbf{v_{j1}} + ... + (-s+z_j)^{k_j-2}\alpha_\mathbf{j}\mathbf{v_{j(k_j-1)}} + (-s+z_j)^{k_j-1}\alpha_\mathbf{j}\mathbf{v_{jk_j}} + (-s+z_j)^{k_j}\mathbf{G_0}(s)]$$

has at least $k_j \times (n-1)$ zeros at z_j. It is deduced that $\det[\mathbf{G_A}(s)]$ has at most k_j zeros at z_j.
Therefore, $\det[\mathbf{G_A}(s)]$ has a zero at z_j of multiplicity k_j

Conversely, suppose that $\mathbf{G_A}(s)$ has a k_j multiplicity zero z_j with the zero direction $\mathbf{v_{jk}}(k = 1, 2, ..., k_j)$. Since $\det[\mathbf{G_A}(s)]$ is not identically zero, $\mathbf{G_A}(s)$ is invertible. $\mathbf{G_A}^{-1}(s)$ has k_j multiplicity poles at z_j

Proof ctd.4.

It has the following Laurent expression:

$$\mathbf{G_A}^{-1}(s) = \sum_{i=-k_j}^{\infty} (-s+z_j)^i \mathbf{R_{i+k_j+1}}$$

$\mathbf{R_i}(i = 1, 2, ..., \infty)$ are $n \times n$ constant matrices. $\mathbf{G_A}(s)$ does not have poles at z_j. Hence, $\mathbf{R_1} \neq \mathbf{0}$.
Choose a nonzero $1 \times n$ vector β so that $\beta\mathbf{R_1} \neq \mathbf{0}$. We have

$$\beta\mathbf{G_A}^{-1}(s) = (-s+z_j)^{-k_j} \sum_{i=-k_j}^{\infty} \beta(-s+z_j)^{i+k_j} \mathbf{R_{i+k_j+1}}$$

Rewrite the equation as

Proof ctd.5.

$$(-s+z_j)^{k_j}\beta = \left[\sum_{i=-k_j}^{\infty} \beta(-s+z_j)^{i+k_j}\mathbf{R_{i+k_j+1}}\right]\mathbf{G_A}(s)$$

Compute the lth ($l = 0, 1, ..., k_j - 1$) derivatives of the two sides at z_j:

$$\lim_{s\to z_j}\frac{d^l}{ds^l}\left\{\left[\sum_{i=-k_j}^{\infty} \beta(-s+z_j)^{i+k_j}\mathbf{R_{i+k_j+1}}\right]\mathbf{G_A}(s)\right\} = 0$$
$$l = 0, 1, ..., k_j - 1$$

which can be reduced to

Proof ctd.6.

$$\lim_{s\to z_j} \frac{d^l}{ds^l}\left\{\left[\sum_{i=-k_j}^{-1} \beta(-s+z_j)^{i+k_j}\mathbf{R_{i+k_j+1}}\right]\mathbf{G_A}(s)\right\}=0$$

$$l=0,1,...,k_j-1$$

Compare it with the definition of zero directions:

$$\lim_{s\to z_j} \frac{d^l}{ds^l}\left\{\left[\sum_{i=-k_j}^{-1} \mathbf{v_{j(i+k_j+1)}}(-s+z_j)^{i+k_j}\right]\mathbf{G_A}(s)\right\}=0$$

$$l=0,1,...,k_j-1$$

Proof ctd.7.

It is easy to see that $\beta \mathbf{R_k}(k = 1, 2, ..., k_j)$ is a multiple of $\mathbf{v_{jk}}$ for any row vector β. This happens if and only if there exists a nonzero column vector $\boldsymbol{\alpha}_\mathbf{j}$ so that $\mathbf{R_k} = \boldsymbol{\alpha}_\mathbf{j}\mathbf{v_{jk}}(k = 1, 2, ..., k_j)$ $\square$

Lemma

If an $n \times n$ stable transfer function matrix $\mathbf{G_A}(s)$ satisfies

1. $\mathbf{G_A}^*(s)\mathbf{G_A}(s) = \mathbf{I}$, *and*
2. z_j *of multiplicity* $k_j(j = 1, 2, ..., r_z)$ *are the only open RHP zeros of* $\mathbf{G_A}(s)$ *with zero directions* $\mathbf{v_{jk}}(k = 1, 2, ..., k_j)$,

then $\mathbf{G_A}(s)$ *is the inner factor of* $\mathbf{G}(s)$ *and* $\mathbf{G_{MP}}(s) = \mathbf{G_A}^{-1}(s)\mathbf{G}(s)$ *is the outer factor of* $\mathbf{G}(s)$

Proof.

It is enough to prove that $\mathbf{G_{MP}}(s)$ is MP and has the same closed RHP poles as $\mathbf{G}(s)$.

Since $z_j (j = 1, 2, ..., r_z)$ are the only open RHP zeros of $\mathbf{G_A}(s)$, the only open RHP poles of $\mathbf{G_A}^{-1}(s)$ are z_j. Recall Lemma 11. If z_j is a k_j multiplicity zero of $\mathbf{G_A}(s)$, $\mathbf{G_A}^{-1}(s)$ has an expression of the form

$$\begin{aligned}\mathbf{G_A}^{-1}(s) &= (-s+z_j)^{-k_j}\boldsymbol{\alpha}_\mathbf{j}\mathbf{v_{j1}} + ... + \\ &\quad (-s+z_j)^{-2}\boldsymbol{\alpha}_\mathbf{j}\mathbf{v_{j(k_j-1)}} + \\ &\quad (-s+z_j)^{-1}\boldsymbol{\alpha}_\mathbf{j}\mathbf{v_{jk_j}} + \mathbf{G_0}(s)\end{aligned}$$

where $\boldsymbol{\alpha}_\mathbf{j}$ is a nonzero column vector and $\mathbf{G_0}(s)$ is a term without poles at z_j

Proof ctd.1.

Multiply both sides on the right by $\mathbf{G}(s)$:

$$\begin{aligned}\mathbf{G_A}^{-1}(s)\mathbf{G}(s) &= (-s+z_j)^{-k_j}\alpha_\mathbf{j}\mathbf{v_{j1}}\mathbf{G}(s)+\ldots+\\ &\quad (-s+z_j)^{-2}\alpha_\mathbf{j}\mathbf{v_{j(k_j-1)}}\mathbf{G}(s)+\\ &\quad (-s+z_j)^{-1}\alpha_\mathbf{j}\mathbf{v_{jk_j}}\mathbf{G}(s)+\mathbf{G_0}(s)\mathbf{G}(s)\end{aligned}$$

To obtain the Laurent expression of $\mathbf{G_A}^{-1}(s)\mathbf{G}(s)$ at z_j, all coefficients of the terms with $(-s+z_j)^{-i}(i=1,2,\ldots,k_j)$ have to be computed. This can be achieved by multiplying both sides by $(-s+z_j)^{-k_j}$, and then computing the lth $(l=0,1,\ldots,k_j-1)$ derivatives at z_j. The computing procedure is similar to that in Lemma of Section 14.3

Proof ctd.2.

By utilizing the definition of zero directions, it can be found that all coefficients are zero for $j = 1, 2, ..., r_z$. In other words, all open RHP poles of $\mathbf{G_A}^{-1}(s)$ are cancelled by those open RHP zeros of $\mathbf{G}(s)$

The implication of this fact is twofold:

1. $\mathbf{G_A}^{-1}(s)$ does not introduce any closed RHP poles to $\mathbf{G_A}^{-1}(s)\mathbf{G}(s)$ $\mathbf{G_{MP}}(s)$ has the same closed RHP poles as $\mathbf{G}(s)$
2. All open RHP zeros of $\mathbf{G}(s)$ are removed. $\mathbf{G_{MP}}(s)$ does not have any open RHP zeros

This completes the proof

□

Theorem

The matrix $\mathbf{G_A}(s)$ *of the form*

$$\mathbf{G_A}(s) = \mathbf{I} - \mathbf{B}^*(s\mathbf{I} + \bar{\mathbf{A}})^{-1}\mathbf{F}^{-1}\mathbf{B}$$

is inner. Here

$$\mathbf{A} = \begin{bmatrix} \mathbf{A_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbf{A_{r_z}} \end{bmatrix}, \mathbf{A_j} = \begin{bmatrix} z_j & -1 & & \\ & z_j & \ddots & \\ & & \ddots & -1 \\ & & & z_j \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{B_1} \\ \vdots \\ \mathbf{B_{r_z}} \end{bmatrix}, \mathbf{B_j} = \begin{bmatrix} \mathbf{v_{j1}} \\ \vdots \\ \mathbf{v_{jk_j}} \end{bmatrix}, j = 1, 2, ..., r_z$$

Theorem (ctd.1)

F *is the solution of the following Lyapunov equation:*

$$\mathbf{F}\bar{\mathbf{A}} + \mathbf{A}^T\mathbf{F} = \mathbf{B}\mathbf{B}^*$$

Proof.

First, it is shown that $\mathbf{G_A}^*(s)\mathbf{G_A}(s) = \mathbf{I}$. By using a similar procedure to the that for the simple zero, it can be proved that

$$\mathbf{G_A}^*(s) = \mathbf{I} - \mathbf{B}^*\mathbf{F}^{-1}(-s\mathbf{I} + \mathbf{A}^T)^{-1}\mathbf{B}$$

Since $\mathbf{F} = \mathbf{F}^*$,

ctd.1.

$$
\begin{aligned}
& \mathbf{G_A}^*(s)\mathbf{G_A}(s) \\
= \;& [\mathbf{I} - \mathbf{B}^*(s\mathbf{I} + \bar{\mathbf{A}})^{-1}\mathbf{F}^{-1}\mathbf{B}]^*[\mathbf{I} - \mathbf{B}^*(s\mathbf{I} + \bar{\mathbf{A}})^{-1}\mathbf{F}^{-1}\mathbf{B}] \\
= \;& [\mathbf{I} - \mathbf{B}^*\mathbf{F}^{-1}(-s\mathbf{I} + \mathbf{A}^T)^{-1}\mathbf{B}][\mathbf{I} - \mathbf{B}^*(s\mathbf{I} + \bar{\mathbf{A}})^{-1}\mathbf{F}^{-1}\mathbf{B}] \\
= \;& \mathbf{I} - \mathbf{B}^*\mathbf{F}^{-1}(-s\mathbf{I} + \mathbf{A}^T)^{-1}\mathbf{B} - \mathbf{B}^*(s\mathbf{I} + \bar{\mathbf{A}})^{-1}\mathbf{F}^{-1}\mathbf{B} + \\
& \mathbf{B}^*\mathbf{F}^{-1}(-s\mathbf{I} + \mathbf{A}^T)^{-1}\mathbf{B}\mathbf{B}^*(s\mathbf{I} + \bar{\mathbf{A}})^{-1}\mathbf{F}^{-1}\mathbf{B} \\
= \;& \mathbf{I} - \mathbf{B}^*\mathbf{F}^{-1}(-s\mathbf{I} + \mathbf{A}^T)^{-1}\{\mathbf{F}(s\mathbf{I} + \bar{\mathbf{A}}) + (-s\mathbf{I} + \mathbf{A}^T)\mathbf{F} - \\
& \mathbf{B}\mathbf{B}^*\}(s\mathbf{I} + \bar{\mathbf{A}})^{-1}\mathbf{F}^{-1}\mathbf{B} \\
= \;& \mathbf{I} - \mathbf{B}^*\mathbf{F}^{-1}(-s\mathbf{I} + \mathbf{A}^T)^{-1}(\mathbf{F}\bar{\mathbf{A}} + \mathbf{A}^T\mathbf{F} - \mathbf{B}\mathbf{B}^*)(s\mathbf{I} + \bar{\mathbf{A}})^{-1}\mathbf{F}^{-1}\mathbf{B}
\end{aligned}
$$

Proof ctd.2.

It is known that $\mathbf{F}$ satisfies the following Lyapunov equation:

$$\mathbf{F}\bar{\mathbf{A}} + \mathbf{A}^T\mathbf{F} = \mathbf{B}\mathbf{B}^*$$

One readily obtains that $\mathbf{G_A}^*(s)\mathbf{G_A}(s) = \mathbf{I}$

Next, examine the zero of $\mathbf{G_A}(s)$. Since

$$\mathbf{G_A}^{-1}(s) = \mathbf{G_A}^*(s) = \mathbf{I} - \mathbf{B}^*\mathbf{F}^{-1}(-s\mathbf{I} + \mathbf{A}^T)^{-1}\mathbf{B}$$

$z_j (j = 1, 2, ..., r_z)$ are the only open RHP zeros of $\mathbf{G_A}(s)$
By Lemma 12 it is concluded that $\mathbf{G_A}(s)$ is inner □

It is easy to verify that the theorem for simple zeros is a special case of the theorem for multiple zeros

14.4 Analysis and Computation

Result in the last three sections: An analytical solution to the extended inner-outer factorization was developed. Provided that the zeros of the plant are given, the factorization can be computed by a formula in closed form

Normally, the zeros of the plant are known for the sake of analyzing the stability or estimating the performance

Computation of zeros: Analytical for low-order plants, but not analytical for high-order plants. The computation of zeros is equivalent to computing the roots of an equation in a single unknown. If the order of the equation is more than 4, then there is **not** analytical formula for the computation

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Computation complexity: When the zeros of the plant are known, the computation complexity of the factorization depends on the multiplicity of the zero in the RHP

As it was seen in preceding sections, to compute the inner matrix, **A**, **B**, and **F** must be obtained first:

1. **A** is from construction. It is exactly known
2. **B** is also from construction, but the zero directions have to be computed first. If the zero directions are known, **B** is exactly known
3. **F** has to be computed on the basis of the zero directions

The zero directions can be computed by the following formulas:

$$
\begin{aligned}
\mathbf{v_{j1}}\mathbf{G}(z_j) &= 0,\\
\mathbf{v_{j2}}\mathbf{G}(z_j) &= \mathbf{v_{j1}}\frac{d}{ds}\mathbf{G}(z_j),\\
\cdots &\quad ,\\
\mathbf{v_{jk_j}}\mathbf{G}(z_j) &= \sum_{i=1}^{k_j-1}(-1)^{k_j-i+1}\frac{\mathbf{v_{ji}}}{(k_j-i)!}\frac{d^{k_j-i}}{ds^{k_j-i}}\mathbf{G}(z_j),\\
j &= 1,2,...,r_z
\end{aligned}
$$

They can directly be derived from the definition of zero direction.

If the plant has only simple open RHP zeros, the zero direction can be obtained by

$$\mathbf{v_{j1}}\mathbf{G}(z_j) = \mathbf{0}, j = 1,2,...,r_z$$

Computation of zero directions: Because $\mathbf{G}(s)$ loses rank at z_j and $\mathbf{v_{j1}}$ can be any nonzero vector satisfying the foregoing formulas, the computation of $\mathbf{v_{j1}}$ is very simple

For example, if $\mathbf{G}(z_j) = [0\ 0; 1\ 2]$, one can simply take $\mathbf{v_{j1}} = [1\ 0]$. By using the foregoing formulas, the computation of $\mathbf{v_{j2}}$ is simple once $\mathbf{v_{j1}}$ is known. For the computation of $\mathbf{v_{jk}}(k > 2)$, the complexity is mainly from the computation of the derivative of $\mathbf{G}(s)$ at z_j

Computation of F: Now it is shown that when the zero directions are exactly known, $\mathbf{F}$ can be exactly computed. $\mathbf{F}$ is the solution of the following Lyapunov equation:

$$\mathbf{F}\bar{\mathbf{A}} + \mathbf{A}^T\mathbf{F} = \mathbf{B}\mathbf{B}^*$$

Express $\mathbf{F}$ as a block matrix:

$$\mathbf{F} = [\mathbf{F_{ij}}], \mathbf{F_{ij}} = \left[f_{xy}^{ij}\right], i, j = 1, 2, ..., r_z; x, y = 1, 2, ..., k_j$$

Let $f_{x0}^{ij} = f_{0y}^{ij} = 0$ for all i, j, x, y. $\mathbf{F}$ can be directly computed by the following formula:

$$f_{xy}^{ij} = \frac{\mathbf{v_{ix}}\mathbf{v_{jy}}^*}{\bar{z}_j + z_i} + \frac{f_{(x-1)y}^{ij} + f_{x(y-1)}^{ij}}{\bar{z}_j + z_i}$$

In particular, when the plant has only simple RHP zeros, we have

$$\mathbf{F} = \left[f_{11}^{ij}\right], f_{11}^{ij} = \frac{\mathbf{v_{i1}}\mathbf{v_{j1}^*}}{\bar{z}_j + z_i}$$

Note that the computation of **F** is an **exact** procedure without any numerical error

The deriving procedure for the factorization formula is very long. However, the result is simple, because it is analytical. The computation is summarized as follows

Given an $n \times n$ transfer function matrix $\mathbf{G}(s)$, its k_j multiplicity open RHP zeros $z_j (j = 1, 2, ..., r_z)$, and zero directions $\mathbf{v_{jk}} (k = 1, 2, ..., k_j)$. The inner matrix can be exactly computed through the following steps:

1. $$\mathbf{A} = \begin{bmatrix} \mathbf{A_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbf{A_{r_z}} \end{bmatrix}, \mathbf{A_j} = \begin{bmatrix} z_j & -1 & & \\ & z_j & \ddots & \\ & & \ddots & -1 \\ & & & z_j \end{bmatrix}$$
2. $$\mathbf{B} = \begin{bmatrix} \mathbf{B_1} \\ \vdots \\ \mathbf{B_{r_z}} \end{bmatrix}, \mathbf{B_j} = \begin{bmatrix} \mathbf{v_{j1}} \\ \vdots \\ \mathbf{v_{jk_j}} \end{bmatrix}$$
3. $\mathbf{F} = [\mathbf{F_{ij}}], \mathbf{F_{ij}} = \left[f_{xy}^{ij} \right], f_{xy}^{ij} = \frac{\mathbf{v_{ix}} \mathbf{v_{jy}}^*}{\bar{z}_j + z_i} + \frac{f_{(x-1)y}^{ij} + f_{x(y-1)}^{ij}}{\bar{z}_j + z_i}$, and $f_{x0}^{ij} = f_{0y}^{ij} = 0$
4. $\mathbf{G_A}(s) = \mathbf{I} - \mathbf{B}^*(s\mathbf{I} + \bar{\mathbf{A}})^{-1}\mathbf{F}^{-1}\mathbf{B}$

Two typical examples are given here to illustrate the use of these formulas

Example

Consider the plant described by the following transfer function matrix:

$$\mathbf{G}(s) = \begin{bmatrix} \frac{(s-1)^2}{(s+1)^2} & \frac{(s-1)^2}{(s+1)^2} \\ \frac{-1}{s+1} & \frac{s-2}{s+1} \end{bmatrix}$$

The plant has a RHP zero of multiplicity 3 at $s = 1$ (that is, $r_z = 1, k_j = 3$) and a LHP pole of multiplicity 3 at $s = -1$. The zero directions can be obtained based on computation formulas:

$$\mathbf{v_{11}} = [1\ 0], \mathbf{v_{12}} = [1\ 0], \mathbf{v_{13}} = [0\ 1/2]$$

Example (ctd.1)

As the open RHP zeros and zero directions are known, it is readily obtained that

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$\mathbf{F}$ can be computed by means of (1) and (1):

$$\mathbf{F} = \begin{bmatrix} 1/2 & 3/4 & 3/8 \\ 3/4 & 5/4 & 13/16 \\ 3/8 & 13/16 & 15/16 \end{bmatrix}$$

Example (ctd.2)

Consequently,

$$
\begin{aligned}
\mathbf{G_A}(s) &= \mathbf{I} - \mathbf{B}^*(s\mathbf{I} + \bar{\mathbf{A}})^{-1}\mathbf{F}^{-1}\mathbf{B} \\
&= \mathbf{I} - \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \left(s\mathbf{I} + \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \right)^{-1} \\
&\quad \begin{bmatrix} 1/2 & 3/4 & 3/8 \\ 3/4 & 5/4 & 13/16 \\ 3/8 & 13/16 & 15/16 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1/2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{5s^3-7s^2-s+3}{5(s+1)^3} & \frac{-4(s-1)^2}{5(s+1)^3} \\ \frac{-4}{5(s+1)} & \frac{5s-3}{5(s+1)} \end{bmatrix}
\end{aligned}
$$

Example (ctd.3)

and

$$\mathbf{G_{MP}}(s) = \begin{bmatrix} \frac{5s+7}{5(s+1)} & \frac{5s+11}{5(s+1)} \\ \frac{-1}{5(s+1)} & \frac{5s+2}{5(s+1)} \end{bmatrix}$$

Since the computation is analytical, the result is exact

Example

Consider the following plant:

$$\mathbf{G}(s) = \frac{1}{s+1}\begin{bmatrix} s-1 & s-1 \\ -1 & s-2 \end{bmatrix}$$

The plant has a RHP zero of multiplicity 2 at $s = 1$ and a LHP pole of multiplicity 2 at $s = -1$

Example (ctd.1)

The zero directions are

$$\mathbf{v_{11}} = [1\ 0], \mathbf{v_{12}} = [1\ \ -1]$$

It is easy to obtain that

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \mathbf{F} = \begin{bmatrix} 1/2 & 3/4 \\ 3/4 & 7/4 \end{bmatrix}$$

Then

$$\begin{aligned} & \mathbf{G_A}(s) \\ = \ & \mathbf{I} - \mathbf{B}^*(s\mathbf{I} + \bar{\mathbf{A}})^{-1}\mathbf{F}^{-1}\mathbf{B} \\ = \ & \mathbf{I} - \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \left(s\mathbf{I} + \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1/4 & 3/4 \\ 3/4 & 7/4 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \end{aligned}$$

Example (ctd.2)

Hence,

$$\mathbf{G_A}(s) = \begin{bmatrix} \frac{5s^2-2s-3}{5(s+1)^2} & \frac{-4(s-1)}{5(s+1)^2} \\ \frac{-4}{5(s+1)} & \frac{5s-3}{5(s+1)} \end{bmatrix}$$

and

$$\mathbf{G_{MP}}(s) = \begin{bmatrix} \frac{5s+7}{5(s+1)} & \frac{5s+11}{5(s+1)} \\ \frac{-1}{5(s+1)} & \frac{5s+2}{5(s+1)} \end{bmatrix}$$

14.5 Solution of the H_2 Optimal Control Problem

Goal of this section: The parameterization in Section 13.1 and the extended inner-outer factorization in Section 14.3 will be used to analytically derive the H_2 optimal controller

H_2 optimal control:

$$\min \|\mathbf{S}(s)\mathbf{W}(s)\|_2$$

where $\mathbf{W}(s) = \mathbf{I}/s$ is the weighting function and

$$\mathbf{S}(s) = \mathbf{I} - \mathbf{G}(s)\mathbf{Q}(s)$$

Here $\mathbf{Q}(s)$ is the IMC controller. When $\mathbf{Q}(s)$ is known, the unity feedback loop controller can be obtained as follows:

$$\mathbf{C}(s) = \mathbf{Q}(s)[\mathbf{I} - \mathbf{G}(s)\mathbf{Q}(s)]^{-1}$$

Lemma

Assume that $\mathbf{G_A}(s)$ is the inner factor of $\mathbf{G}(s)$.
${\mathbf{G_A}}^{-1}(s) - {\mathbf{G_A}}^{-1}(0)$ *has only unstable poles*

Proof.

It has been known that

$${\mathbf{G_A}}^{-1}(s) = \mathbf{I} - \mathbf{B}^*\mathbf{F}^{-1}(-s\mathbf{I} + \mathbf{A}^T)^{-1}\mathbf{B}$$

$\mathbf{G_A}(0)$ is a constant matrix. It does not affect the distribution of poles. From the expression of ${\mathbf{G_A}}^{-1}(s)$, it is known that ${\mathbf{G_A}}^{-1}(s) - {\mathbf{G_A}}^{-1}(0)$ has only unstable poles □

In Section 13.1, all stabilizing controllers with asymptotic tracking properties are parameterized. Substitute the parameterization into the H_2 optimization problem. We have:

$$\begin{aligned} & \left\| s^{-1}\mathbf{S}(s) \right\|_2^2 \\ = & \left\| s^{-1}[\mathbf{I} - \mathbf{G}(s)\mathbf{Q}(s)] \right\|_2^2 \\ = & \left\| s^{-1}\{\mathbf{I} - \mathbf{G}(s)\mathbf{G}^{-1}(0)[\mathbf{I} + s\mathbf{Q_1}(s)]\} \right\|_2^2 \end{aligned}$$

Theorem

Assume that the plant can be factored into two parts:

$$\mathbf{G}(s) = \mathbf{G_A}(s)\mathbf{G_{MP}}(s),$$

where $\mathbf{G_A}(s)$ *is the inner factor given by Theorem 14 and* ${\mathbf{G_{MP}}}^{-1}(s)$ *is the corresponding outer factor. Then the unique optimal solution for the* H_2 *optimal control is*

$$\mathbf{Q_{opt}}(s) = {\mathbf{G_{MP}}}^{-1}(s){\mathbf{G_A}}^{-1}(0)$$

Proof.

Since $\mathbf{G_A}^*(s)\mathbf{G_A}(s) = I$, we have

$$
\begin{aligned}
& \left\| s^{-1}\mathbf{S}(s) \right\|_2^2 \\
= & \left\| \ \mathbf{G_A}(s)s^{-1}\{\mathbf{G_A}^{-1}(s) - \mathbf{G_{MP}}(s)\mathbf{G}^{-1}(0)[\mathbf{I} + s\mathbf{Q_1}(s)]\} \ \right\|_2^2 \\
= & \left\| \ s^{-1}\{\mathbf{G_A}^{-1}(s) - \mathbf{G_{MP}}(s)\mathbf{G}^{-1}(0)[\mathbf{I} + s\mathbf{Q_1}(s)]\} \ \right\|_2^2 \\
= & \left\| \begin{array}{l} s^{-1}[\mathbf{G_A}^{-1}(s) - \mathbf{G_A}^{-1}(0)] + \\ s^{-1}\{\mathbf{G_A}^{-1}(0) - \mathbf{G_{MP}}(s)\mathbf{G}^{-1}(0)[\mathbf{I} + s\mathbf{Q_1}(s)]\} \end{array} \right\|_2^2
\end{aligned}
$$

s is a factor of

$$\mathbf{G_A}^{-1}(s) - \mathbf{G_A}^{-1}(0)$$

Proof ctd.1.

Since $\mathbf{G_{MP}}(0)\mathbf{G}^{-1}(0) = \mathbf{G_A}^{-1}(0)$, s must be a factor of

$$\mathbf{G_A}^{-1}(0) - \mathbf{G_{MP}}(s)\mathbf{G}^{-1}(0)[\mathbf{I} + s\mathbf{Q_1}(s)].$$

It is evident that

$$s^{-1}[\mathbf{G_A}^{-1}(s) - \mathbf{G_A}^{-1}(0)]$$

is strictly proper.

$$s^{-1}\{\mathbf{G_A}^{-1}(0) - \mathbf{G_{MP}}(s)\mathbf{G}^{-1}(0)[\mathbf{I} + s\mathbf{Q_1}(s)]\}$$

is also strictly proper if $\mathbf{Q}(s) = \mathbf{G}^{-1}(0)[\mathbf{I} + s\mathbf{Q_1}(s)]$ is proper

Proof ctd.2.

On the other hand, it was proved that

$$s^{-1}[\mathbf{G_A}^{-1}(s) - \mathbf{G_A}^{-1}(0)]$$

has only unstable poles.

$$s^{-1}\{\mathbf{G_A}^{-1}(0) - \mathbf{G_{MP}}(s)\mathbf{G}^{-1}(0)[\mathbf{I} + s\mathbf{Q_1}(s)]\}$$

is stable. To see this, let us consider the following equality:

$$\begin{aligned} & \mathbf{G_A}(s)\{\mathbf{G_A}^{-1}(0) - \mathbf{G_{MP}}(s)\mathbf{G}^{-1}(0)[\mathbf{I} + s\mathbf{Q_1}(s)]\} \\ = \; & [\mathbf{G_A}(s)\mathbf{G_A}^{-1}(0) - \mathbf{I}] + \{\mathbf{I} - \mathbf{G}(s)\mathbf{G}^{-1}(0)[\mathbf{I} + s\mathbf{Q_1}(s)]\} \end{aligned}$$

Proof ctd.3.

As $\mathbf{G_A}(s)$ is stable, the first term in the right-hand side is stable. By Theorem for parameterization, the second term is stable, too. To find the optimal controller, a constrained search will be replaced with an unconstrained one in the design procedure. Temporarily relax the constraint on $\mathbf{Q}(s)$. We have

$$\begin{aligned} & \left\| s^{-1}\mathbf{S}(s) \right\|_2^2 \\ = & \left\| s^{-1}[\mathbf{G_A}^{-1}(s) - \mathbf{G_A}^{-1}(0)] \right\|_2^2 + \\ & \left\| s^{-1}\{\mathbf{G_A}^{-1}(0) - \mathbf{G_{MP}}(s)\mathbf{G}^{-1}(0)[\mathbf{I} + s\mathbf{Q_1}(s)]\} \right\|_2^2 \end{aligned}$$

Minimizing the right-hand side of the equation, we obtain that

$$\mathbf{Q_{1opt}}(s) = s^{-1}[\mathbf{G}(0)\mathbf{G_{MP}}^{-1}(s)\mathbf{G_A}^{-1}(0) - \mathbf{I}]$$

Proof ctd.4.

A little algebra yields

$$\mathbf{Q}_{\mathbf{opt}}(s) = \mathbf{G}_{\mathbf{MP}}{}^{-1}(s)\mathbf{G}_{\mathbf{A}}{}^{-1}(0)$$

□

It can be seen that the controller order is directly related to the plant order. Since the optimal solution is obtained, the achievable performance can be directly estimated

Corollary

The optimal performance for the H_2 control is

$$\left\| s^{-1}\mathbf{B}^*\mathbf{F}^{-1}[(\mathbf{A}^T)^{-1} - (-s\mathbf{I} + \mathbf{A}^T)^{-1}]\mathbf{B} \right\|_2$$

Proof.

$$
\begin{aligned}
& \mathbf{G_A}^{-1}(s) - \mathbf{G_A}^{-1}(0) \\
= \; & [\mathbf{I} - \mathbf{B}^*\mathbf{F}^{-1}(-s\mathbf{I} + \mathbf{A}^T)^{-1}\mathbf{B}] - [\mathbf{I} - \mathbf{B}^*\mathbf{F}^{-1}(\mathbf{A}^T)^{-1}\mathbf{B}] \\
= \; & \mathbf{B}^*\mathbf{F}^{-1}(\mathbf{A}^T)^{-1}\mathbf{B} - \mathbf{B}^*\mathbf{F}^{-1}(-s\mathbf{I} + \mathbf{A}^T)^{-1}\mathbf{B} \\
= \; & \mathbf{B}^*\mathbf{F}^{-1}[(\mathbf{A}^T)^{-1} - (-s\mathbf{I} + \mathbf{A}^T)^{-1}]\mathbf{B}
\end{aligned}
$$

Therefore,

$$
\min \left\| s^{-1}\mathbf{S}(s) \right\|_2 = \left\| s^{-1}\mathbf{B}^*\mathbf{F}^{-1}[(\mathbf{A}^T)^{-1} - (-s\mathbf{I} + \mathbf{A}^T)^{-1}]\mathbf{B} \right\|_2
$$

□

14.6 Filter Design

The optimal controller $\mathbf{Q_{opt}}(s)$ is usually improper. To implement the controller, a filter $\mathbf{J}(s)$ must be introduced

Suboptimal control: When the plant model is exactly known, the optimal solution can be arbitrarily approximated by choosing an appropriate filter while the internal stability is kept

The optimal control: The optimal solution can never be reached, as the optimal controller is physically irrealizable

In general, the filter should satisfy the following requirements:

1. The controller $\mathbf{Q}(s) = \mathbf{Q_{opt}}(s)\mathbf{J}(s)$ is proper
2. The closed-loop system is internally stable
3. Asymptotic tracking is achieved

For clarity of presentation, it is assumed that the system inputs are steps. Depending on different plants, the filter is chosen in different ways.

Stable plants

For stable plants, the filter can be chosen as a diagonal one:

$$\mathbf{J}(s) = \begin{bmatrix} J_1(s) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & J_n(s) \end{bmatrix}$$

with

$$J_i(s) = \frac{1}{(\lambda_i s + 1)^{n_i}}, i = 1, 2, ..., n$$

where $\lambda_i (i = 1, 2, ..., n)$ are performance degrees

The fist condition is easy to satisfy. Assume that the largest relative degree in any element of the ith column of $\mathbf{Q_{opt}}(s)$ is α_i. To satisfy the first condition, one can take $n_i = \alpha_i$ for strictly proper columns and $n_i = 1$ for semi-proper columns

The second condition and the third condition are already satisfied. Since $\mathbf{J}(s)$ is stable, the closed-loop system must be internally stable. $\mathbf{J}(0) = \mathbf{I}$. Hence,

$$\lim_{s\to 0} \det[\mathbf{I} - \mathbf{G}(s)\mathbf{Q}(s)] = \lim_{s\to 0} \det[\mathbf{I} - \mathbf{G_A}(s)\mathbf{G_A}^{-1}(0)\mathbf{J}(s)] = 0$$

The tuning method for quantitative performance and robustness is similar to that in the H_2 decoupling control system. As different loops are decoupled, the tuning procedure is more complex than that for decoupling control

Unstable MP plants

For unstable MP plants, the filter can also be chosen as a diagonal one:

$$\mathbf{J}(s) = \begin{bmatrix} J_1(s) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & J_n(s) \end{bmatrix}$$

with

$$J_i(s) = \frac{N_{xi}(s)}{(\lambda_i s + 1)^{n_i}}, i = 1, 2, ..., n$$

where $\lambda_i (i = 1, 2, ..., n)$ are performance degrees, $N_{xi}(s)$ are polynomials with all roots in the LHP and $N_{xi}(0) = 1$. Suppose l_{ij} is the largest multiplicity of the unstable pole $p_j (j = 1, 2, ..., r_p)$ in the ith row of $\mathbf{G}(s)$. $\deg\{N_{xi}(s)\} = l_{ij}$

Assume that the largest relative degree in any element of the ith column of $\mathbf{Q}_{\mathbf{opt}}(s)$ is α_i. To satisfy the first condition, one can take $n_i = \deg\{N_{xi}(s)\} + \alpha_i$ for strictly proper columns and $n_i = \deg\{N_{xi}(s)\} + 1$ for semi-proper columns

For MP plants, $\mathbf{G}(s)\mathbf{Q}_{\mathbf{opt}}(s) = \mathbf{I}$. The closed-loop response is decoupled. To make the closed-loop system internally stable, the ith element of the filter should satisfy

$$\lim_{s \to p_j} \frac{d^k}{ds^k}[1 - J_i(s)] = 0$$

$$i = 1, 2, ..., n; j = 1, 2, ..., r_p; k = 0, 1, ..., l_{ij} - 1$$

Since $\mathbf{J}(0) = \mathbf{I}$, the third condition is already satisfied

Unstable NMP plants

For unstable NMP plants, a more complex structure may be necessary for the filter

The fist condition is easy to satisfy. As it is known, an improper transfer function implies that the degree of its numerator is greater than that of its denominator. To make it proper, a pole-zero excess should be introduced by the filter. This is not difficult.

The second condition is normally not easy to satisfy, because $\mathbf{J}(s)$ is determined by

$$\lim_{s \to p_j} \frac{d^k}{ds^k} \det[\mathbf{I} - \mathbf{G_A}(s)\mathbf{G_A}^{-1}(0)\mathbf{J}(s)] = 0$$

$$j = 1, 2, ..., r_p; k = 0, 1, ..., l_j - 1$$

To solve the problem, let $\mathbf{J}(s) = \mathbf{J_F}(s)\mathbf{J_D}(s)$, where the subscripts F and D denote "full matrix" and "diagonal matrix", respectively

$$\begin{aligned}\mathbf{J_F}(s) &= \mathbf{G_A}(0)\mathbf{G_A}^{-1}(s)\\ \mathbf{J_D}(s) &= \begin{bmatrix} J_1(s) & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & J_n(s)\end{bmatrix}\end{aligned}$$

with

$$J_i(s) = \prod_{j=1}^{r_z}(-s/z_j+1)^{k_{ij}}\frac{N_{xi}(s)}{(\lambda_i s+1)^{n_i}}, i = 1, 2, ..., n$$

where $\lambda_i (i = 1, 2, ..., n)$ are performance degrees, $N_{xi}(s)$ are polynomials with all roots in the LHP, and k_{ij} is the largest multiplicity of z_j in the ith column of $\mathbf{J_F}(s)$

As $\mathbf{J_D}(s)$ removes all unstable poles of $\mathbf{J_F}(s)$, $\mathbf{J}(s)$ is stable. In this case, the second condition reduces to

$$\lim_{s \to p_j} \frac{d^k}{ds^k}[1 - J_i(s)] = 0$$

$$i = 1, 2, ..., n; j = 1, 2, ..., r_p; k = 0, 1, ..., l_{ij} - 1$$

The third condition can be achieved by choosing $N_{xi}(0) = \mathbf{I}$. The order of $\mathbf{J_D}(s)$ should be chosen such that $\mathbf{Q}(s)$ is proper

As $\mathbf{T}(s) = \mathbf{G}(s)\mathbf{Q}(s) = \mathbf{J_D}(s)$, the obtained response is decoupled. As a matter of fact, the response is identical to that in Section 12.2

Compared to the design with the weighting functions $\mathbf{W_{p1}}(s)$ and $\mathbf{W_{p2}}(s)$, the introduction of filter simplifies the design task. The designer is not required to choose the weighting function by trial and error

Now let us see how to simplify the selection of the weighting functions when using a filter

Consider the weighting function $\mathbf{W_{p1}}(s)$ first. In Section 10.4, it is assumed that the inputs are unit steps, and the controller is designed only for pre-specified weight $\mathbf{W_{p1}}(s) = s^{-1}\mathbf{I}$ and the following performance index:

$$\min \|\mathbf{W_{p2}}(s)\mathbf{S}(s)\mathbf{W_{p1}}(s)\|_2^2$$

However, the system inputs may be complicated

Compared to the design with the weighting functions $\mathbf{W_{p1}}(s)$ and $\mathbf{W_{p2}}(s)$, the introduction of filter simplifies the design task. The designer is not required to choose the weighting function by trial and error

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Consider the weighting function $\mathbf{W_{p1}}(s)$ first. In Section 10.4, it is assumed that the inputs are unit steps, and the controller is designed only for pre-specified weight $\mathbf{W_{p1}}(s) = s^{-1}\mathbf{I}$ and the following performance index:

$$\min \|\mathbf{W_{p2}}(s)\mathbf{S}(s)\mathbf{W_{p1}}(s)\|_2^2$$

However, the system inputs may be complicated

They may be steps with lags (for example, $r(s) = 1/s/(s+1)$) or the ramp (that is, $r(s) = 1/s/s$). If the weight function $\mathbf{W}_{p1}(s)$ is chosen to equal the input, the design procedure will be complex

In this case, one can choose the weighting function as $s^{-1}\mathbf{I}$ and adopt the following simple design procedure:

1. Design the controller for unit steps
2. Choose an appropriate filter $\mathbf{J}(s)$ to satisfy the constraints imposed by asymptotic properties

The design procedure can be used for both the optimal control and the decoupling control

Now consider the weighting function $\mathbf{W}_{p2}(s)$. As it is known, $\mathbf{W}_{p2}(s)$ is used to weight errors over different frequency ranges

Although the optimal controller can be derived for a general weighting function $\mathbf{W}_{p2}(s)$, the procedure and the obtained controller will be complicated

The optimal solution for general $\mathbf{W}_{p2}(s)$ is

$$\begin{aligned}\mathbf{Q_{opt}}(s) \;=\;& \mathbf{G_{MP}}^{-1}(s)[\mathbf{W_{p2}}(s)\mathbf{G_A}(s)]_{\mathbf{MP}}^{-1} \\ & \{s\{[\mathbf{W_{p2}}(s)\mathbf{G_A}(s)]_{\mathbf{MP}}\mathbf{G_A}^{-1}(s)/s - \\ & [\mathbf{W_{p2}}(0)\mathbf{G_A}(0)]_{\mathbf{MP}}\mathbf{G_A}^{-1}(0)/s\}_* + \\ & [\mathbf{W_{p2}}(0)\mathbf{G_A}(0)]_{\mathbf{MP}}\mathbf{G_A}^{-1}(0)\}\end{aligned}$$

where $\mathbf{W_{p2}}(s)\mathbf{G_A}(s) = [\mathbf{W_{p2}}(s)\mathbf{G_A}(s)]_{\mathbf{A}}[\mathbf{W_{p2}}(s)\mathbf{G_A}(s)]_{\mathbf{MP}}$, $[\mathbf{W_{p2}}(s)\mathbf{G_A}(s)]_{\mathbf{A}}$ and $[\mathbf{W_{p2}}(s)\mathbf{G_A}(s)]_{\mathbf{MP}}$ denote the all-pass and MP parts of $\mathbf{W_{p2}}(s)\mathbf{G_A}(s)$, respectively. $[\mathbf{W_{p2}}(0)\mathbf{G_A}(0)]_{\mathbf{MP}}$ denotes the value of $[\mathbf{W_{p2}}(s)\mathbf{G_A}(s)]_{\mathbf{MP}}$ at $s = 0$. $\{\cdot\}_*$ denotes that after a partial fraction expansion of the function all terms involving the poles z_j are removed

This is why the controller is designed only for a simple weighting function $\mathbf{W}_{p2}(s) = \mathbf{I}$. With the help of the filter, the errors can be weighted in an easy way; that is, the weighting is achieved by tuning

14.7 Examples for designing H_2 Optimal Controllers

The purpose of this section is to demonstrate the H_2 optimal design procedure. The design procedure is summarized as follows:

1. Factorize the plant: $\mathbf{G}(s) = \mathbf{G_A}(s)\mathbf{G_{MP}}(s)$, where $\mathbf{G_A}(s) = \mathbf{I} - \mathbf{B}^*(s\mathbf{I} + \bar{\mathbf{A}})^{-1}\mathbf{F}^{-1}\mathbf{B}$
2. Compute the optimal controller: $\mathbf{Q_{opt}}(s) = \mathbf{G_{MP}}^{-1}(s)\mathbf{G_A}^{-1}(0)$
3. Introduce the filter to the optimal controller: $\mathbf{Q}(s) = \mathbf{Q_{opt}}(s)\mathbf{J}(s)$. The unity feedback controller is $\mathbf{C}(s) = \mathbf{Q}(s)\left[\mathbf{I} - \mathbf{G}(s)\mathbf{Q}(s)\right]^{-1}$

Three examples are provided in this section:

1. In the first example, the controller is analytically designed and tuned for the required quantitative undershoot
2. The second example is used to illustrate the quantitative tuning on weighting errors of different channels
3. In the third example, a real plant with frequency domain design requirement is considered. It is shown how the quantitative requirement can be easily achieved by means of the design method introduced in this chapter

Example

Consider the following plant:

$$\boxed{\mathbf{G}(s) = \frac{1}{(s+1)^3}\begin{bmatrix} (s-1)^2 & (s-1)^2 \\ (s-1)(s-2) & 2(s-1)(s-2) \end{bmatrix}}$$

The plant has three NMP zeros at $s = 1$, one NMP zero at $s = 2$, and 6 stable poles at $s = -1$. One zero at $s = 1$ is the common zero of all elements of $\mathbf{G}(s)$. As introduced in Section 14.3, the first step is to separate the common zero as follows:

$$\frac{-s+1}{s+1}$$

The next step is to factorize the remainder of $\mathbf{G}(s)$

Example (ctd.1)

Let

$$\mathbf{G_r}(s) = \frac{s+1}{-s+1}\mathbf{G}(s) = \frac{-1}{(s+1)^2}\begin{bmatrix} s-1 & s-1 \\ s-2 & 2(s-2) \end{bmatrix}$$

Since

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{F} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/4 \end{bmatrix}$$

The inner factor of $\mathbf{G_r}(s)$ is

$$\mathbf{G_A}(s) = \begin{bmatrix} \frac{s-1}{s+1} & 0 \\ 0 & \frac{s-2}{s+2} \end{bmatrix}$$

Example (ctd.2)

The inner factor of the original plant $\mathbf{G}(s)$ is

$$\frac{-s+1}{s+1}\mathbf{G_A}(s) = \frac{-s+1}{s+1}\begin{bmatrix} \frac{s-1}{s+1} & 0 \\ 0 & \frac{s-2}{s+2} \end{bmatrix}$$

Therefore

$$\mathbf{G_{MP}}(s) = \frac{s+1}{-s+1}\mathbf{G_A}^{-1}(s)\mathbf{G}(s) = \frac{-1}{(s+1)^2}\begin{bmatrix} s+1 & s+1 \\ s+2 & 2(s+2) \end{bmatrix}.$$

It is easy to verify that for the special plant, the H_2 optimal control and the H_2 decoupling control result in the same factorization

Example (ctd.3)

Using Theorem in Section 14.5, the optimal controller is

$$\mathbf{Q}_{\mathbf{opt}}(s) = \mathbf{G}_{\mathbf{MP}}{}^{-1}(s)\mathbf{G}_{\mathbf{A}}{}^{-1}(0) = \frac{s+1}{s+2}\begin{bmatrix} 2(s+2) & -(s+1) \\ -(s+2) & s+1 \end{bmatrix}$$

The plant is stable. For step inputs, choose

$$\mathbf{J}(s) = \begin{bmatrix} \frac{1}{\lambda_1 s+1} & 0 \\ 0 & \frac{1}{\lambda_2 s+1} \end{bmatrix}$$

The suboptimal controller is

$$\mathbf{Q}(s) = \mathbf{Q}_{\mathbf{opt}}(s)\mathbf{J}(s) = \frac{s+1}{s+2}\begin{bmatrix} \frac{2(s+2)}{\lambda_1 s+1} & \frac{-(s+1)}{\lambda_2 s+1} \\ \frac{-(s+2)}{\lambda_1 s+1} & \frac{s+1}{\lambda_2 s+1} \end{bmatrix}$$

Example (ctd.4)

The sensitivity function is

$$\begin{aligned}\mathbf{S}(s) &= \mathbf{I} - \mathbf{G}(s)\mathbf{Q}(s) \\ &= \begin{bmatrix} 1 - \frac{(s-1)^2}{(s+1)^2(\lambda_1 s+1)} & 0 \\ 0 & 1 - \frac{(s-1)(s-2)}{(s+1)(s+2)(\lambda_2 s+1)} \end{bmatrix}\end{aligned}$$

The unity feedback loop controller is

$$\boxed{\mathbf{C}(s) = \begin{bmatrix} \frac{2(s+1)^3}{s(\lambda_1 s^2+2\lambda_1 s+\lambda_1+4)} & \frac{-(s+1)^3}{s(\lambda_2 s^2+3\lambda_2 s+2\lambda_2+6)} \\ \frac{-(s+1)^3}{s(\lambda_1 s^2+2\lambda_1 s+\lambda_1+4)} & \frac{(s+1)^3}{s(\lambda_2 s^2+3\lambda_2 s+2\lambda_2+6)} \end{bmatrix}}$$

This controller is exact. There is not any numerical error

Example (ctd.5)

The performance degrees are determined by the desired closed-loop response, such as overshoot, robustness, the shape of $\mathbf{S}(s)$, and so on. Suppose the design specification is 20% undershoot with the shortest rise time for both of the two loops. One can take $\lambda_1 = 1.25$ and $\lambda_2 = 1.05$. The closed-loop responses are shown in Figures

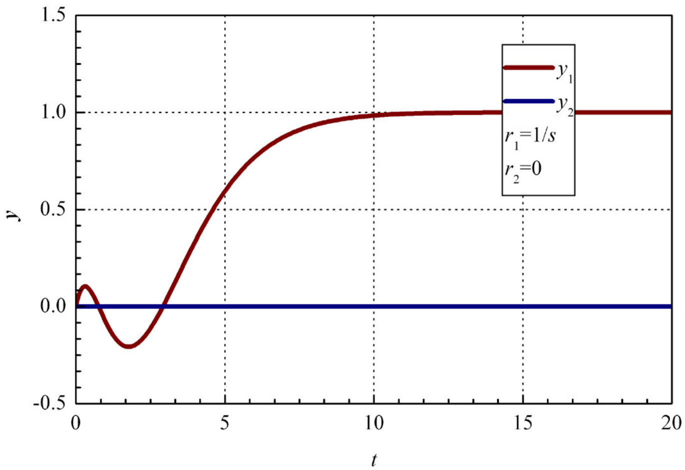


Figure: Response of the system with $\lambda_1 = 1.25$ and $\lambda_2 = 1.05$-1

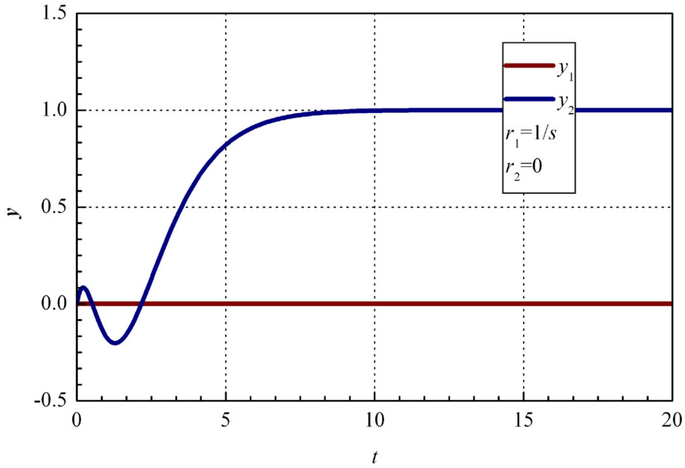


Figure: Response of the system with $\lambda_1 = 1.25$ and $\lambda_2 = 1.05$-2

Example

Consider the plant in Example of Section 14.4:

$$\boxed{\mathbf{G}(s) = \frac{1}{s+1}\begin{bmatrix} s-1 & s-1 \\ -1 & s-2 \end{bmatrix}}$$

It has been obtained that

$$\mathbf{G_A}(s) = \begin{bmatrix} \frac{5s^2-2s-3}{5(s+1)^2} & \frac{-4(s-1)}{5(s+1)^2} \\ \frac{-4}{5(s+1)} & \frac{5s-3}{5(s+1)} \end{bmatrix}$$

and

$$\mathbf{G_{MP}}(s) = \begin{bmatrix} \frac{5s+7}{5(s+1)} & \frac{5s+11}{5(s+1)} \\ \frac{-1}{5(s+1)} & \frac{5s+2}{5(s+1)} \end{bmatrix}$$

Example (ctd.1)

Hence, the optimal controller is

$$\mathbf{Q}_{\mathbf{opt}}(s) = \mathbf{G}_{\mathbf{MP}}{}^{-1}(s)\mathbf{G}_{\mathbf{A}}{}^{-1}(0) = \frac{1}{5(s+1)}\begin{bmatrix} -(7s+10) & -(s-5) \\ 4s+5 & -(3s+5) \end{bmatrix}$$

Introduce the following filter for step inputs:

$$\mathbf{J}(s) = \begin{bmatrix} \frac{1}{\lambda_1 s+1} & 0 \\ 0 & \frac{1}{\lambda_2 s+1} \end{bmatrix}$$

The suboptimal controller is

$$\mathbf{Q}(s) = \mathbf{Q}_{\mathbf{opt}}(s)\mathbf{J}(s) = \frac{1}{5(s+1)}\begin{bmatrix} \frac{-(7s+10)}{\lambda_1 s+1} & \frac{-(s-5)}{\lambda_2 s+1} \\ \frac{4s+5}{\lambda_1 s+1} & \frac{-(3s+5)}{\lambda_2 s+1} \end{bmatrix}$$

Example (ctd.2)

The sensitivity function is

$$\begin{aligned} \mathbf{S}(s) &= \mathbf{I}-\mathbf{G}(s)\mathbf{Q}(s) \\ &= \begin{bmatrix} 1-\frac{5s^2-2s-3}{5(s+1)^2(\lambda_1 s+1)} & \frac{4(s-1)}{5(s+1)^2} \\ \frac{4}{5(s+1)(\lambda_1 s+1)} & 1-\frac{5s-3}{5(s+1)(\lambda_2 s+1)(\lambda_1 s+1)} \end{bmatrix} \end{aligned}$$

The unity feedback loop controller is

$$\mathbf{C}(s)=\mathbf{Q}(s)[\mathbf{I}-\mathbf{G}(s)\mathbf{Q}(s)]$$

Suppose that the design specification is $y_2 < 0.3$ for $r_1 = 1/s$ and $r_2 = 0$, $y_1 < 0.3$ for $r_1 = 0$ and $r_2 = 1/s$. One can take $\lambda_1 = 1$ and $\lambda_2 = 0.5$

Example (ctd.3)

The controller is

$$\boxed{\mathbf{C}(s) = \frac{1}{s(5s^2+34s+65)} \begin{bmatrix} -(7s^2+41s+34) & -2(s^2-8s-9) \\ 4s^2+17s+17 & -2(3s^2+16s+13) \end{bmatrix}}$$

The closed-loop responses are shown in Figures

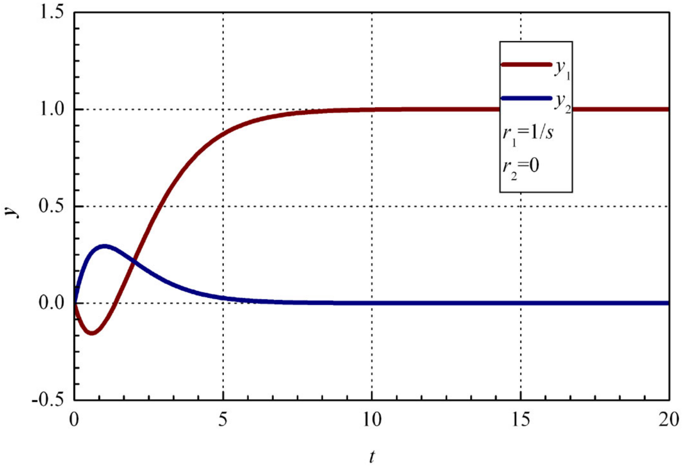


Figure: Response of the system with $\lambda_1 = 1$ and $\lambda_2 = 0.5$-1

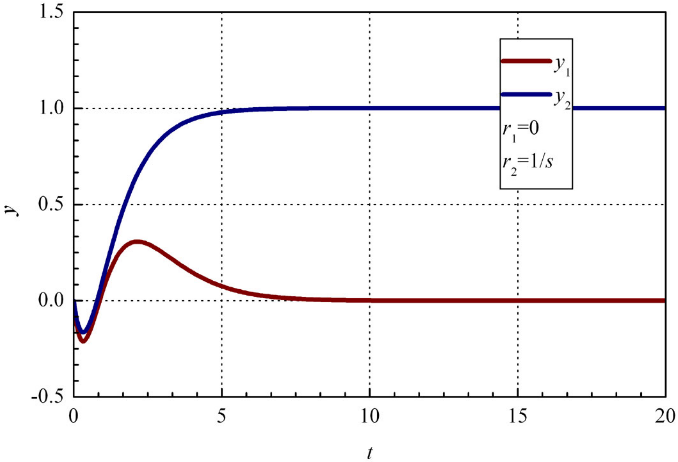
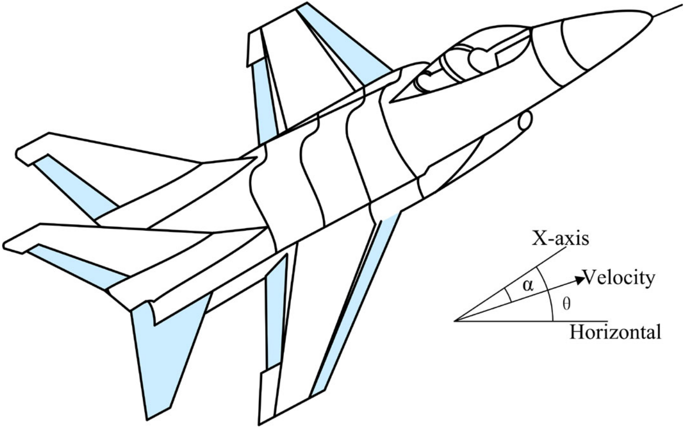


Figure: Response of the system with $\lambda_1 = 1$ and $\lambda_2 = 0.5$-2

Example

The longitudinal dynamics of an aircraft trimmed at 25 000ft and 0.9 Mach is unstable and has two RHP phugoid modes



Example (ctd.1)

The linear model can be expressed in the form of

$$\mathbf{G}(s) = \frac{\begin{bmatrix} n_{11}(s) & n_{12}(s) \\ n_{21}(s) & n_{22}(s) \end{bmatrix}}{d(s)}$$

where

$$\begin{aligned}
n_{11}(s) &= -5.1240s^4 - 1099.4s^3 - 28390s^2 - 568.48s + 24.076 \\
n_{12}(s) &= -948.12s^3 - 30325s^2 - 56482s - 1215.3, \\
n_{21}(s) &= -0.14896s^4 + 655.67s^3 + 19817s^2 + 385.44s - 61.970 \\
n_{22}(s) &= 671.88s^3 + 21446s^2 + 38716s + 916.45, \\
d(s) &= s^6 + 64.554s^5 + 1167.0s^4 + 3728.6s^3 - \\
& \quad 5495.4s^2 + 1102.0s + 708.10
\end{aligned}$$

Example (ctd.2)

The singular value design specification is:

1. Robustness specification:-40 dB/decade roll-off and at least -20 dB at 100 rad/sec.
2. Performance specification: Minimize the sensitivity function as much as possible.

The plant is unstable and MP. There are two unstable poles at $s = 0.6898 + 0.2488i$ and $s = 0.6898 - 0.2488i$. The optimal solution is

$$\mathbf{Q}_{opt}(s) = \mathbf{G}^{-1}(s)$$

The largest relative degree of the first column is -2, the largest relative degree of the second column is -3, and the plant has two unstable poles

Example (ctd.3)

The following filter is chosen:

$$\mathbf{J}(s) = \begin{bmatrix} \frac{\beta_{12}s^2+\beta_{11}s+1}{(\lambda_1 s+1)^4} & 0 \\ 0 & \frac{\beta_{22}s^2+\beta_{21}s+1}{(\lambda_2 s+1)^5} \end{bmatrix}$$

With the following constraints:

$$\lim_{s\to 0.6898+0.2488\mathrm{i}} [1 - J_i(s)] = 0, \lim_{s\to 0.6898-0.2488\mathrm{i}} [1 - J_i(s)] = 0, i = 1, 2$$

we have

$$\begin{aligned} \beta_{12} &= -10.2624\lambda_1 + 6\lambda_1^2 - 0.5377\lambda_1^4 + \\ & \quad 1.3796(7.4387\lambda_1 - 4\lambda_1^3 + 1.3796\lambda_1^4) \\ \beta_{11} &= 0.5377(7.4387\lambda_1 - 4\lambda^3 + 1.3796\lambda_1^4) \end{aligned}$$

Example (ctd.3)

$$\begin{aligned}
\beta_{22} &= -12.828\lambda_2 + 10\lambda_2^2 - 2.6887\lambda_2^4 + 0.7419\lambda_2^5 + \\
&\quad 1.3796(9.2984\lambda_2 - 10\lambda_2^3 + 6.898\lambda_2^4 - 1.3656\lambda_2^5) \\
\beta_{21} &= 0.53773(9.2984\lambda_2 - 10\lambda_2^3 + 6.898\lambda_2^4 - 1.3656\lambda_2^5)
\end{aligned}$$

Therefore, the closed loop transfer function matrix is

$$\mathbf{T}(s) = \begin{bmatrix} \frac{\beta_{12}s^2+\beta_{11}s+1}{(\lambda_1 s+1)^4} & 0 \\ 0 & \frac{\beta_{22}s^2+\beta_{21}s+1}{(\lambda_2 s+1)^5} \end{bmatrix}$$

and the sensitivity function matrix is

$$\mathbf{S}(s) = \mathbf{I} - \begin{bmatrix} \frac{\beta_{12}s^2+\beta_{11}s+1}{(\lambda_1 s+1)^4} & 0 \\ 0 & \frac{\beta_{22}s^2+\beta_{21}s+1}{(\lambda_2 s+1)^5} \end{bmatrix}$$

Example (ctd.4)

It is seen that the closed-loop response is thoroughly decoupled. Since the relative degrees of the two loops in the system are greater than 2, the singular value satisfies the specification of -40 dB/decade roll-off. For simplicity, let the two performance degrees be the same. Increase the performance degrees until reaching -20 dB at 100 rad/sec. The performance degrees are $\lambda_1 = \lambda_2 = 0.16$. The closed-loop responses are shown in Figure

Once the critical $\mathbf{T}(s)$ is determined, $\mathbf{S}(s)$ is determined at the same time owing to the constraint $\mathbf{T}(s) + \mathbf{S}(s) = \mathbf{I}$

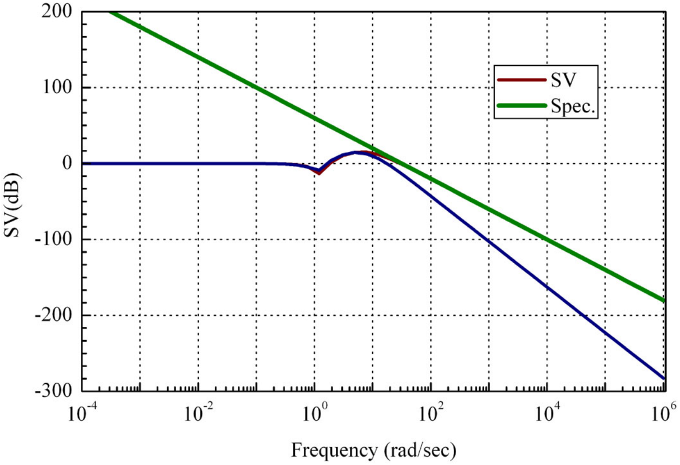


Figure: Response of the system with $\lambda_1 = \lambda_2 = 0.16$.

End of Chapter 14