Chapter 4 H_∞ PID Controllers for Stable Plants

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4.1 Traditional Design Methods

PID Controllers

Importance: 95% controllers in practice are PID controllers
Ideal PID:

$$u(t) = K_c \left[e(t) + \frac{1}{T_I} \int e(t)dt + T_D \frac{de(t)}{dt} \right]$$

K_c—Gain
T_I—Integral constant
T_D—Derivative constant

$e(t)$—Error
$u(t)$—Controller output

Assume that $C(s)$ is the transfer function from $e(s)$ to $u(s)$. Using the Laplace transform, we have

$$C(s) = K_c \left(1 + \frac{1}{T_I s} + T_D s \right)$$

Practical PID Forms

Ideal PID: Has a pure differentiator in it and therefore is not physically realizable

An important method for realizing an improper transfer function: Introduce a low-pass transfer function to it

Three practical forms:

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\begin{aligned}
C(s) &= K_c\left(1+\frac{1}{T_I s}+\frac{T_D s}{T_F s+1}\right)\\
C(s) &= K_c\left(1+\frac{1}{T_I s}\right)\frac{T_D s+1}{T_F s+1}\\
C(s) &= K_c\left(1+\frac{1}{T_I s}+T_D s\right)\frac{1}{T_F s+1}
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Usually $T_F = 0.1T_D$ in a PID

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Tuning Rules

Assume that the step response model is

$$G(s) = \frac{K}{\tau s + 1} e^{-\theta s}$$

and the ultimate cylce model is K_u and T_u

Table: Frequently used tuning methods.

Tuning methods	R-C method	C-C method	Z-N method
KK_c	$1.2(\theta/\tau)^{-1}$	$1.35(\theta/\tau)^{-1} + 0.27$	$0.6KK_u$
T_I/τ	$2(\theta/\tau)$	$\frac{2.5(\theta/\tau)[1 + (\theta/\tau)/5]}{1 + 0.6(\theta/\tau)}$	$0.5T_u/\tau$
T_D/τ	$0.5(\theta/\tau)$	$\frac{0.37(\theta/\tau)}{1 + 0.2(\theta/\tau)}$	$0.125T_u/\tau$

RZN Tuning

Z-N method: The most widely used method
Disadvantage: The resulting PID controller usually gives excessive overshoot

A solution: Refined Z-N (RZN) method. Perhaps the most famous improved method
The modified PID controller is

$$u(t) = K_c \left\{ [\beta r(t) - y(t)] + \frac{1}{\mu T_I} \int e(t)dt + T_D \frac{de(t)}{dt} \right\}$$

β and μ are determined by extensive simulation studies. Define

Normalized gain: $K_n = KK_u$, Normalized time delay: $\theta_n = \theta/\tau$

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Normalized gain: $K_n = KK_u$, Normalized time delay: $\theta_n = \theta/\tau$

When $2.25 < K_n < 15$ and $0.16 < \theta_n < 0.57$, for a 10% overshoot:

$$\beta = \frac{15 - K_n}{15 + K_n}, \mu = 1$$

and for a 20% overshoot:

$$\beta = \frac{36}{27 + 5K_n}, \mu = 1$$

If $1.5 < K_n < 2.25$ and $0.57 < \theta_n < 0.96$,

$$\beta = \frac{8}{17}\left(\frac{4}{9}K_n + 1\right), \mu = \frac{4}{9}K_n$$

Z-N/RZN usually gives very bad response for the plant with large time delay. Hence, some designers believe that PID cannot be used for the plant with large time delay. Actually, with proper design methods PID **can be** applied to such systems

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4.2 H$_\infty$ PID Controllers for the First-Order Plant

New Design Method

Traditional design method: The control structure is first fixed to be a PID and then the parameters are determined by empirical tuning rules

New design method: An optimal performance index is first defined, and then both the PID control structure and parameters are analytically derived

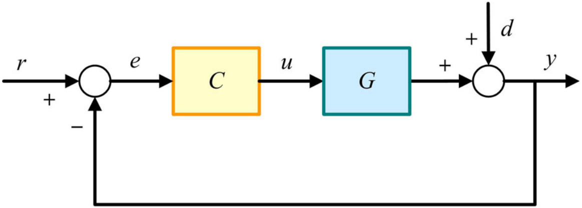

Figure: Unity feedback control loop

Consider the unity feedback control system. According to the Youla parameterization, all stabilizing controllers can be expressed as

$$C(s) = \frac{Q(s)}{1 - G(s)Q(s)}$$

where $Q(s)$ is a stable transfer function. If the model is exact, the transfer function from $d(s)$ to $y(s)$ is given by

$$S(s) = 1 - G(s)Q(s)$$

Take the performance index as

$$\min \|W(s)S(s)\|_\infty$$

where $W(s)$ is a weighting function. It should be chosen so that the 2-norm boundary of the system input is normalized by unity

It is impossible to design a controller for any inputs. Assume that the input is a step, i.e. $d(s) = 1/s$. One can take $W(s) = 1/s$. Consider the first-order plant with time delay:

$$G(s) = \frac{Ke^{-\theta s}}{\tau s + 1}$$

Many plants can be described by the model. With the 1/1 Pade approximant

$$e^{-\theta s} \approx \frac{1 - \theta s/2}{1 + \theta s/2}$$

the approximate plant is

$$G(s) \approx K \frac{1 - \theta s/2}{(\tau s + 1)(1 + \theta s/2)}$$

Basic idea: Design the controller for the approximate plant and then use it for the original plant

Theorem (Maximum Modulus Theorem)

Assume that Ω is a nonempty region in the complex plane and $F(s)$ is a function that does not have poles in Ω. If $F(s)$ is not a constant, then $|F(s)|$ does not attain its maximum value at an interior point of Ω.

Assume that Ω equals the open RHP. $W(s)S(s)$ should not have poles in Ω. By Theorem we have

$$\begin{aligned}\|W(s)S(s)\|_\infty &= \|W(s)[1-G(s)Q(s)]\|_\infty \\ &= \sup_{\mathrm{Re}s>0} |W(s)[1-G(s)Q(s)]|\end{aligned}$$

$G(s)$ has a zero at $s = 2/\theta$ in the open RHP. Accordingly

$$\sup_{\mathrm{Re}s>0} |W(s)[1-G(s)Q(s)]| \geq |W(s)[1-G(s)Q(s)]|_{s=2/\theta}| = \theta/2$$

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There exist two constraints on $Q(s)$:

1. $Q(s)$ should be **stable** for internal stability
2. To make the controller physically realizable, $Q(s)$ should be **proper**
3. To have a finite ∞-norm, $Q(s)$ should satisfy

$$\lim_{s\to 0} S(s) = \lim_{s\to 0}[1 - G(s)Q(s)] = 0$$

This constraint is also required for **asymptotic tracking**

Idea: Drop the requirement of properness first and find the optimal $Q(s)$, namely $Q_{opt}(s)$. Then roll $Q_{opt}(s)$ off at high frequencies

The minimum of $\|W(s)S(s)\|_\infty$ is $\theta/2$. This gives the following unique optimal solution:

$$Q_{opt}(s) = \frac{W(s) - \theta/2}{W(s)G(s)} = \frac{(\tau s + 1)(1 + \theta s/2)}{K}$$

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$Q_{opt}(s)$ is improper. A low-pass filter must be introduced to roll $Q_{opt}(s)$ off at high frequencies. Choose the following filter:

$$J(s) = \frac{\beta_0}{(\lambda s + 1)^2}$$

β_0—A constant
λ—A positive real number
The filter should not violate the constraint for the asymptotic property:

$$\lim_{s \to 0} [1 - G(s) Q_{opt}(s) J(s)] = 0$$

Elementary computations give $\beta_0 = 1$. Then the suboptimal proper $Q(s)$ is

$$Q(s) = Q_{opt}(s) J(s) = \frac{(\tau s + 1)(1 + \theta s/2)}{K(\lambda s + 1)^2}$$

λ is an adjustable parameter called **performance degree**. It closely relates to the closed-loop performance:

$$
\begin{aligned}
\text{Smaller } \lambda \quad & <=> \text{ Fast response} \\
\text{Larger } \lambda \quad & <=> \text{ Slow response} \\
\lambda \to 0 \quad & <=> \text{ The optimal } \|W(s)S(s)\|_\infty
\end{aligned}
$$

The controller of the corresponding unity feedback loop is

$$
C(s) = \frac{Q(s)}{1 - G(s)Q(s)} = \frac{1}{K} \frac{(\tau s + 1)(1 + \theta s/2)}{\lambda^2 s^2 + (2\lambda + \theta/2)s}
$$

This is a PID controller

An important feature: It cancels two poles of the approximate model, or equivalently, two dominant poles of the original model

Compare the H_∞ PID controller with the practical PID controller of the form

$$C(s) = K_C\left(1+\frac{1}{T_I s}+T_D s\right)\frac{1}{T_F s+1}$$

Parameters of the H_∞ PID controller are

$$T_F = \frac{\lambda^2}{2\lambda+\theta/2}, T_I = \frac{\theta}{2}+\tau, T_D = \frac{\theta\tau}{2T_I}, K_C = \frac{T_I}{K(2\lambda+\theta/2)}$$

If the practical PID is in the form of

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In practice, a low-order controller is preferred to a high-order controller. There are two ways to obtain a low-order controller:

1. Design a controller for the high-order model and then reduce the order of the resulting controller
2. Reduce the order of the model and then design a controller

This section adopts the latter

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4.3 H_∞ PID controller and the Smith Predictor

For a very long time, the PID controller and the Smith predictor had been regarded as two irrelevant methods: The Smith predictor was an efficient scheme for plants with large time delays while the PID controller was not. In this section, the internal relationship between the two controllers will be discussed

Assume that $\tilde{G}(s)$ is the real plant, its model is described by

$$G(s) = G_o(s)e^{-\theta s}$$

where $G_o(s)$ is the delay-free part of $G(s)$. When the model is exact and there is no disturbance, the system output is

$$y(s) = C(s)G_o(s)e^{-\theta s}e(s)$$

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This signal is delayed, whereas the desired feedback signal is

$$y_o(s) = C(s)G_o(s)e(s)$$

This is possible if $R(s)$ is substituted for $C(s)$ and the following quantity is added to the open-loop response $y(s)$:

$$y_s(s) = R(s)G_o(s)e(s) - R(s)G_o(s)e^{-\theta s}e(s)$$

since

$$y_s(s) + y(s) = y_o(s)$$

The implication of adding $y_s(s)$ to the signal $y(s)$ is shown in Figure. It is seen that $y_s(s)$ is generated by introducing a simple local loop

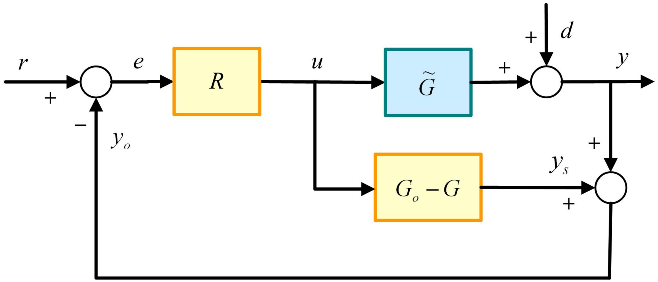

Figure: Structure of the Smith predictor

$R(s)$ differs from $C(s)$ in the unity feedback loop:

$$C(s) = \frac{R(s)}{1 + [G_o(s) - G(s)]R(s)}$$

If the plant is rational, $R(s)$ reduces to $C(s)$

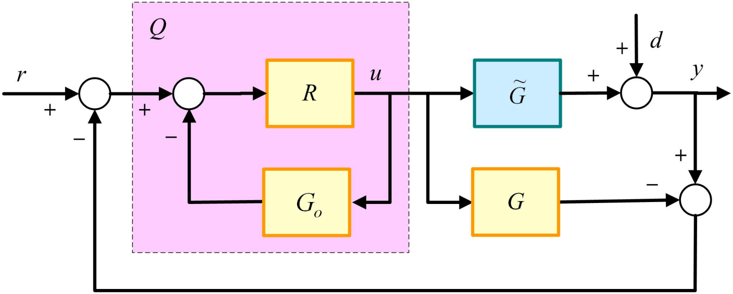

Figure: Equivalent structure of the Smith predictor

It is seen in the figure that $R(s)$ and $Q(s)$ are related through

$$Q(s) = \frac{R(s)}{1 + G_o(s)R(s)}$$

For the H_∞ PID controller given in the last section we have

$$Q(s) = \frac{(\tau s + 1)(1 + \theta s/2)}{K(\lambda s + 1)^2}$$

The controller of the Smith predictor can be obtained by the inverse relationship:

$$R(s) = \frac{Q(s)}{1 - G_o(s)Q(s)} = \frac{1}{K}\frac{(\tau s + 1)(1 + \theta s/2)}{\lambda^2 s^2 + (2\lambda - \theta/2)s}$$

$R(s)$ is a PID controller when $\lambda > \theta/4$

Conclusions

The Smith predictor and the PID controller are approximately equivalent.This implies that the PID controller can also be used to control plants with large time delays, provided it is appropriately designed

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Example

Consider the paper-making machine shown in Figure. The paper-making machine is divided into five sections: head, table and pressing, drying, calenderstack, and reel. Not shown on this figure is the stock preparation system. In this system, fibers are dispersed in water, various other materials in the paper-making suspension are added, and the suspension is delivered to the mixing tank. In the mixing tank and the head-box, the thick stock is mixed with the recycled water. Then the head-box delivers the diluted suspension of fibers to the wire with small fine holes. The wire continuously moves over the table where most of the water is removed by draining through the wire. This produces a wet mat of fibers on the wire, which will become a finished sheet of paper after pressed and dried

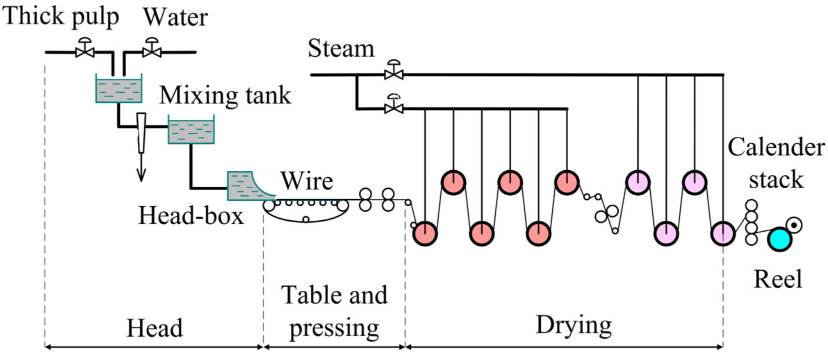


Figure: Paper-making process (From Zhang et al., 2001. Reprinted by permission of the John Wiley & Sons)

In the system, there are many control objectives, of which the most important is basis weight

Example (ctd.1)

By mechanistic analysis and identification, a low-order model has been developed for basis weight control:

$$\boxed{G(s) = \frac{5.15}{1.8s+1}e^{-2.8s}}$$

That is, $K = 5.15, \tau = 1.8, \theta = 2.8$. Then the H_∞ controller is

$$Q(s) = \frac{(1.8s+1)(1.4s+1)}{5.15(\lambda s+1)^2}$$

A little algebra yields the following PID controller:

$$\boxed{C(s) = \frac{(1.8s+1)(1.4s+1)}{5.15[\lambda^2 s^2 + (2\lambda + 1.4)s]}}$$

Example (ctd.2)

The equivalent Smith predictor is

$$R(s) = \frac{(1.8s+1)(1.4s+1)}{5.15[\lambda^2 s^2 + (2\lambda - 1.4)s]}$$

Nominal response: Take $\lambda = 0.4\theta$. A unit step reference is added at $t = 0$ and a step load (that is, the disturbance at the plant input) with the magnitude -0.1 is added at $t = 50$. The nominal response of the closed-loop system is shown in Figure. It is seen that the response of the system is fast and steady.

Robust performance: Assume that there exists 50% error in estimating θ, that is, θ varies in [1.4, 4.2]. Figure shows the system response. If the performance degree is $\lambda = 0.7\theta$, the response becomes slightly slower, but a better robustness is obtained

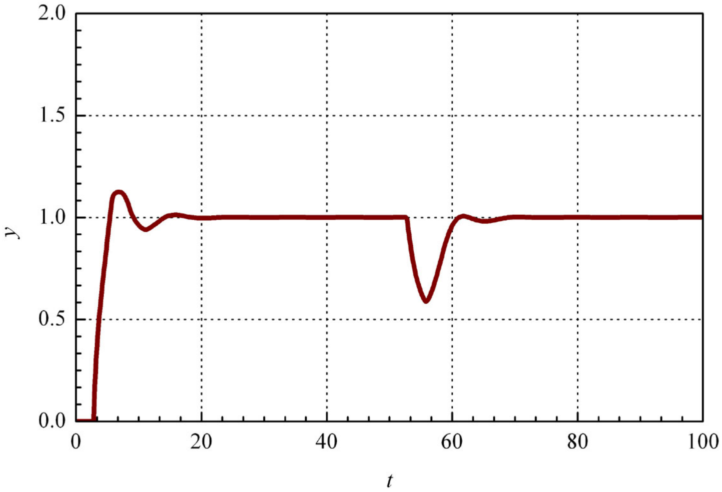


Figure: Nominal response of the closed-loop system

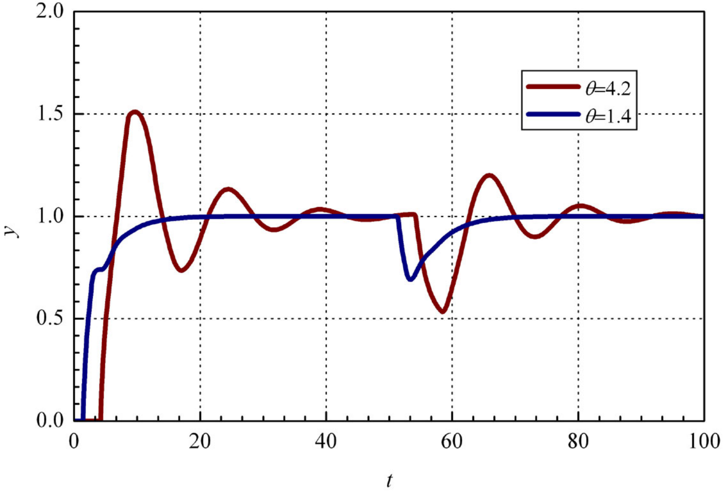
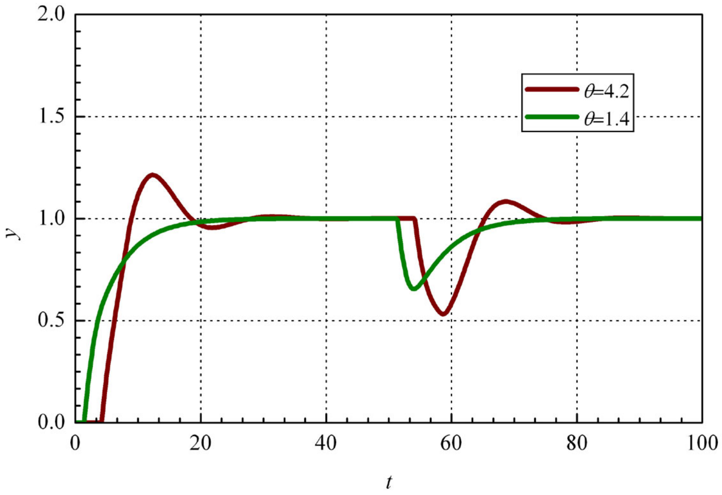


Figure: Response of the uncertain system with $\lambda = 0.4\theta$



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4.4 Quantitative Performance and Robustness

Goal: Show how a quantitative performance or robustness can be obtained by adjusting the performance degree

Case 1: If the real plant were the **approximate model**, the closed-loop transfer function of the system would be

$$T(s) = \frac{1-\theta s/2}{(\lambda s+1)^2}$$

The disturbance transfer function of the system would be

$$S(s) = \frac{\lambda^2 s^2 + (2\lambda + \theta/2)s}{(\lambda s+1)^2}$$

In this case, the performance degree can be freely selected. When $\lambda \to 0$, the system tends to be optimal: $\|W(s)S(s)\|_\infty \to \theta/2$

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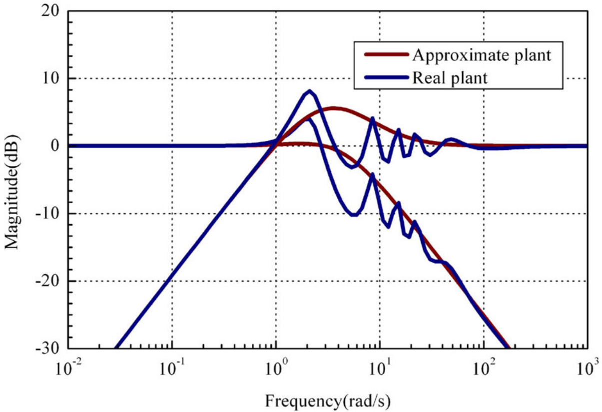

Figure: Frequency response of the closed-loop system

The system with the approximate plant had a steady responses

Case 2:

In the last section, the real plant is in the form of the **first-order plant with time delay** and the Pade approximation was used to treat the time delay. When the obtained controller is applied to the real plant, the response of the closed-loop system fluctuates near the break frequency, which is caused by the error from the Pade approximation.

Regard the error as a kind of known uncertainty and let

$$|\Delta_m(j\omega)| \geq K \left| \frac{e^{-\theta j\omega}}{\tau j\omega + 1} - \frac{1 - \theta j\omega/2}{(\tau j\omega + 1)(1 + \theta j\omega/2)} \right|$$

The robust stability of the closed-loop system can be tested by

$$\|\Delta_m(s) T(s)\|_\infty < 1$$

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$$\|\Delta_m(s) T(s)\|_\infty < 1$$

The performance degree relates the stability and the performance of the closed-loop system in a **monotonous** manner:

1. When the performance degree decreases, $|T(j\omega)|$ increases in the higher frequency range and $|S(j\omega)|$ decreases in the lower frequency range. Such a system has a larger bandwidth. This implies better performance and poor robustness.
2. When the performance degree increases, $|T(j\omega)|$ decreases in the higher frequency range and $|S(j\omega)|$ increases in the lower frequency range. The system has a smaller bandwidth, and thus performance is sacrificed for robustness.

The nominal performance and the robustness of a system conflict with each other. By choosing an appropriate performance degree, one can easily trade off between the nominal performance and the robustness. The monotonicity of the performance degree makes the trade-off procedure, or the controller tuning procedure, very simple

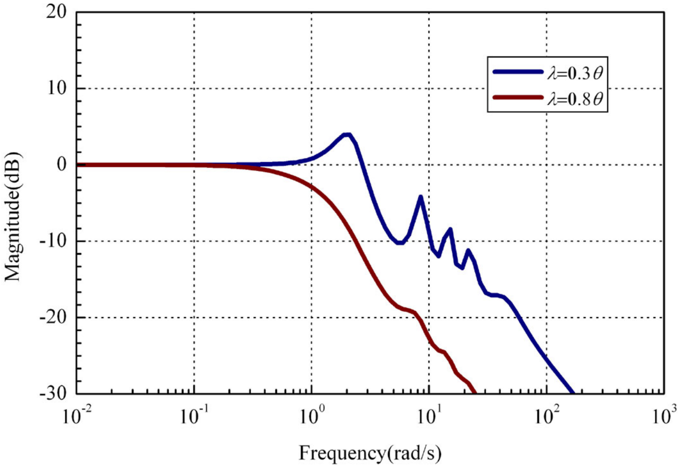

Figure: Relationship between the closed-loop frequency response and λ

Will the Design Cause Stability Problems?

In Section 2.2, an example is given to illustrate that the direct use of rational approximations for stability analysis may lead to an incorrect result. There exists a possibility that the controller stabilizes the approximate model, but cannot stabilize the original model.

Solution in the new design method: The controller is designed for the approximate model, and then used for the original model. That is, the approximate model is regarded as the nominal plant and the approximate error is regarded as the uncertainty. The existence of the approximate error imposes a lower bound on the performance degree for stability. As long as the performance degree is greater than the lower bound, the closed-loop system is stable. The lower bound is about 0.0735θ

Will the Design Cause Stability Problems?

In Section 2.2, an example is given to illustrate that the direct use of rational approximations for stability analysis may lead to an incorrect result. There exists a possibility that the controller stabilizes the approximate model, but cannot stabilize the original model.

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Case 3:

A frequently encountered situation is that the plant is **uncertain**:

$$\widetilde{G}(s) = \frac{\widetilde{K} e^{-\widetilde{\theta} s}}{\widetilde{\tau} s + 1}$$

Then the uncertainty profile is

$$\Delta_m(\omega) \geq \left| \frac{\widetilde{K} e^{-\widetilde{\theta} j\omega}}{\widetilde{\tau} j\omega + 1} - \frac{K(1 - \theta j\omega/2)}{(\tau j\omega + 1)(1 + \theta j\omega/2)} \right|$$

which consists of two parts: The approximate error and the real uncertainty. Then the closed-loop system is stable if and only if

$$\|\Delta_m(s) T(s)\|_\infty < 1$$

This implies that the robust stability can almost always be guaranteed by increasing the performance degree

Tuning for Quantitative Responses

As stated in Section 3.5, there are two classes of design specifications

Case 1: The design specification involving the requirement on robustness is given for the nominal system. In this case, only the nominal performance is considered
In the design **in Section 4.2**, the error introduced by the Pade approximation is clear. Hence, the performance degree has a definite effect on the nominal performance. With the help of numerical methods it can be obtained easily

An important feature: All indices are closely relates to λ/θ

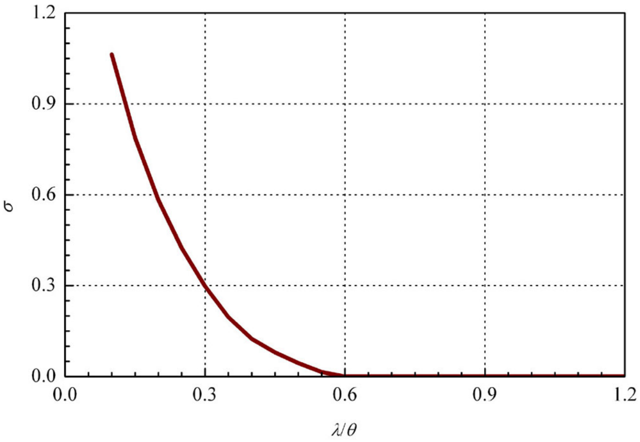

Figure: Effect of the performance degree on the overshoot

e.g., 5% overshoot $->\lambda = 0.5\theta$

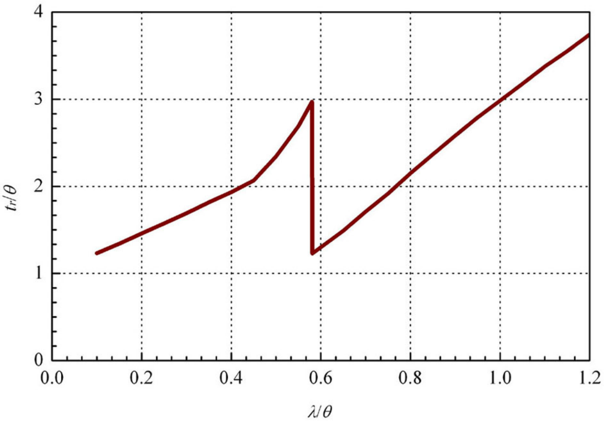

Figure: Effect of the performance degree on the rise time

The sudden change is due to the different definitions for systems with overshoots and without overshoots

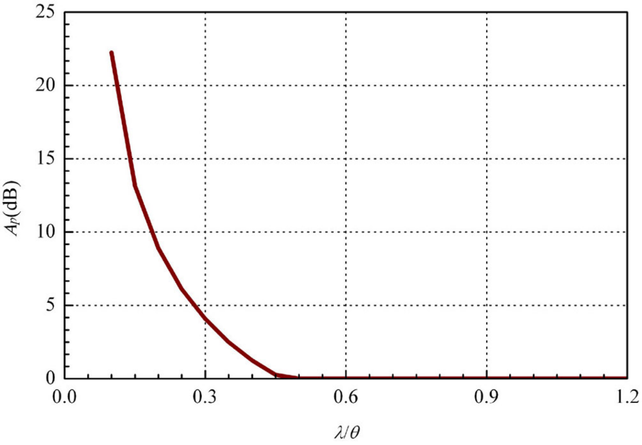

Figure: Effect of the performance degree on the resonance peak

e.g., 2dB $->\lambda = 0.37\theta$

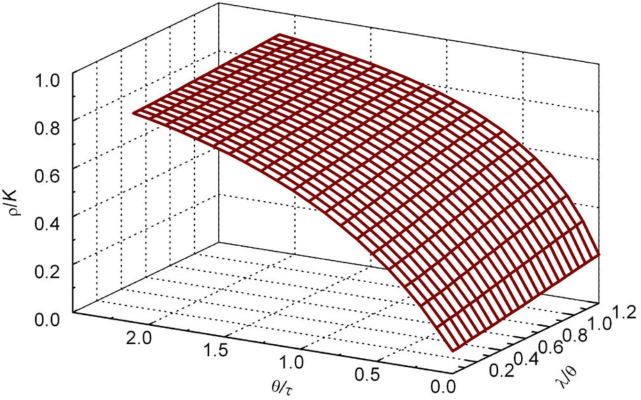

Figure: Effect of the performance degree on the perturbation peak

The disturbance response relates not only to λ/θ, but also to θ/τ

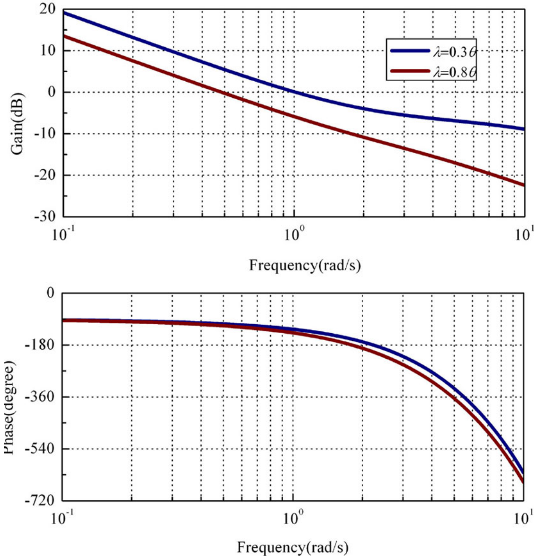

Figure: Bode plot of the H_∞ PID control system

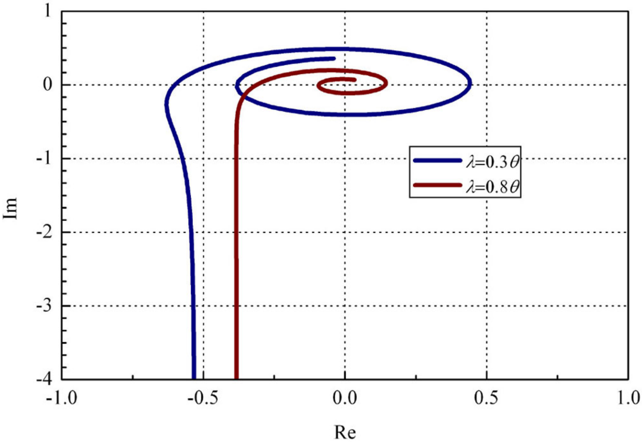

Figure: Nyquist plot of the H_∞ PID control system

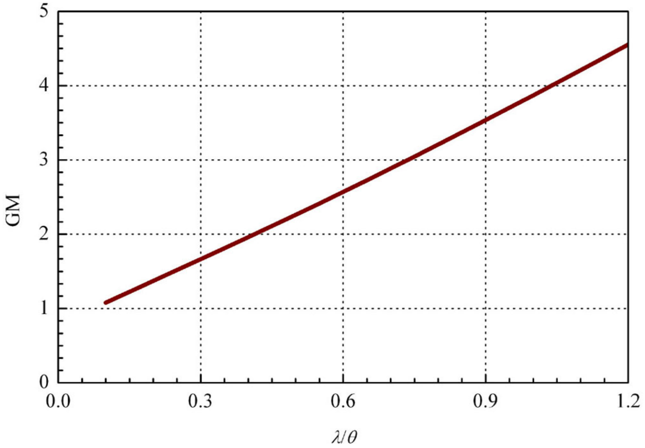

Figure: Effect of the performance degree on the gain margin

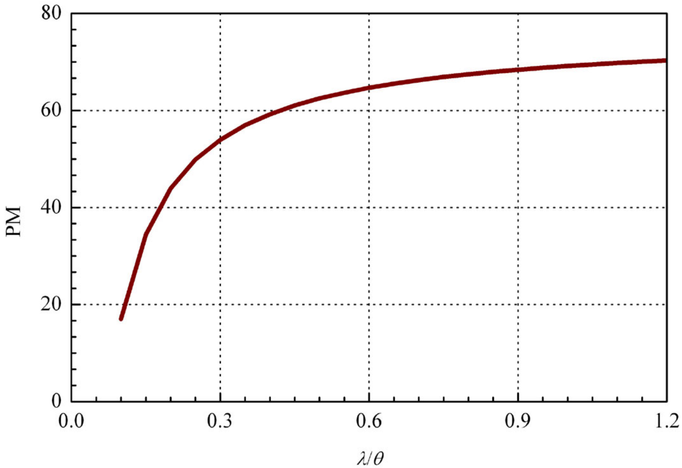

Figure: Effect of the performance degree on the phase margin

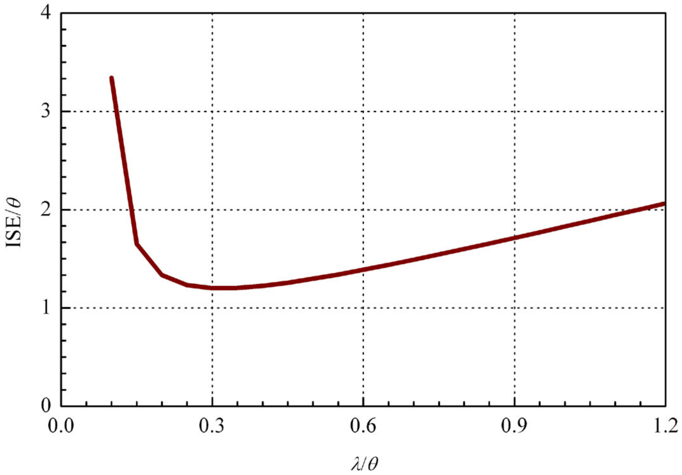

Figure: Effect of the performance degree on ISE

Quantitative Tuning for Robust Performance

Case 2: The design specification is given for the uncertain system. Then, there exists uncertainty in addition to the approximate error

Exact tuning: If the uncertainty profile is obtained, an exact performance degree can be calculated by utilizing the sufficient and necessary condition for robust performance
Problems: The uncertainty profile is not always exactly known due to technical and economic reasons. Even if the uncertainty profile is available, the calculation is involved

A simple tuning method: Without loss of generality, assume that the specification is that the closed-loop system has an overshoot less than 5% in any case, that is, the worst case overshoot is 5%

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The tuning procedure is as follows:

1. Design the controller for the nominal plant. For a 5% overshoot, $\lambda = 0.5\theta$.
2. Substitute the nominal plant by the worst case plant (that is, the gain and the time delay take their maximum value and the time constant takes its minimum value).
3. Increase the performance degree monotonically with a small step until the overshoot equals 5%.

The first step can be omitted. In this case, the initial value of the performance degree is set to be 0. A typical step is 0.01θ or smaller. If the time delay is very small, for instance, $\theta \leq 0.1\tau$, the time constant can be used to determine the step. For example, the step can be taken as 0.01τ or smaller

General Tuning Rules

Both the nominal performance and the robust performance can be quantitatively tuned through such a procedure: **Increase the performance degree monotonically until the required response is obtained**

Recommendation

In many cases, the performance degree can be chosen within the range $0.1\theta - 1.2\theta$. A slightly conservative performance degree is recommended for a real system. Although the response is slower, the system can tolerate larger uncertainty

4.5 H$_\infty$ PID Controllers for the Second-Order Plant

Section 4.2: Utilize 1/1 **Pade approximant** to design PID controllers for the **first-order** plant with time delay

This section: Utilize the first-order **Taylor Series expansion** to design PID controllers for the **second-order** plant with time delay

Assume that the plant model is

$$G(s) = \frac{Ke^{-\theta s}}{(\tau_1 s + 1)(\tau_2 s + 1)}$$

where τ_1 and τ_2 are two time constants. With the first-order Taylor series expansion, we have

$$e^{-\theta s} \approx 1 - \theta s$$

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$$e^{-\theta s} \approx 1 - \theta s$$

The approximate model is

$$G(s) \approx \frac{K(1-\theta s)}{(\tau_1 s+1)(\tau_2 s+1)}$$

Take the performance index as

$$\min \|W(s)S(s)\|_\infty$$

Assume that the system input is a unit step. Then $W(s) = 1/s$. By Maximum Modulus Theorem and the definition of the ∞-norm

$$\|W(s)S(s)\|_\infty = \|W(s)[1-G(s)Q(s)]\|_\infty \geq |W(1/\theta)|$$

for all $Q(s)$s. Minimizing the left-hand side of the equation yields:

$$\left\| \frac{1}{s}\left[1-\frac{K(1-\theta s)}{(\tau_1 s+1)(\tau_2 s+1)}Q(s)\right]\right\|_\infty = \theta$$

It is now clear that the unique optimal solution is

$$Q_{opt}(s) = \frac{(\tau_1 s + 1)(\tau_2 s + 1)}{K}$$

The degree of the numerator polynomial of $Q_{opt}(s)$ is higher by two than that of the denominator polynomial. Since the asymptotic property requires that

$$\lim_{s \to 0}[1 - G(s)Q(s)] = 0$$

the following filter is introduced to roll $Q_{opt}(s)$ off:

$$J(s) = \frac{1}{(\lambda s + 1)^2}$$

A proper $Q(s)$ is then obtained:

$$Q(s) = Q_{opt}(s)J(s) = \frac{(\tau_1 s + 1)(\tau_2 s + 1)}{K(\lambda s + 1)^2}$$

The controller of the unity feedback loop is:

$$C(s) = \frac{Q(s)}{1 - G(s)Q(s)} = \frac{1}{K}\frac{(\tau_1 s + 1)(\tau_2 s + 1)}{\lambda^2 s^2 + (2\lambda + \theta)s}$$

This is a PID controller. If it is realized in the form of

$$C(s) = K_C\left(1 + \frac{1}{T_I s} + T_D s\right)\frac{1}{T_F s + 1}$$

controller parameters are as follows:

$$T_F = \frac{\lambda^2}{2\lambda + \theta}, \quad T_I = \tau_1 + \tau_2,$$
$$T_D = \frac{\tau_1 \tau_2}{\tau_1 + \tau_2}, \quad K_C = \frac{\tau_1 + \tau_2}{K(2\lambda + \theta)}.$$

Usually λ is chosen within the range $0.2\theta - 1.2\theta$

The H_∞ PID controller of the second-order plant possesses similar features to that of the first-order plant. Since there are two time constants in the plant model, the relationship between the performance degree and the system response is complicated.

The roots of the plant are at $-1/\tau_1$ and $-1/\tau_2$:

- If both $1/\tau_1$ and $1/\tau_2$ are positive real, one can reduce the model to the first-order one and then design the controller
- When $1/\tau_1$ and $1/\tau_2$ are conjugate imaginary roots, the controller should be designed for the second-order model

The nominal performance and the robust performance can also be quantitatively tuned through the procedure given in Section 4.4:

Increase the performance degree monotonically until the required response is obtained

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Example

The task of a heat exchanger is to transfer heat from one flow of medium to another. Heat transfer takes place through the thermally conductive material used to separate the two media, one cold and the other hot. Figure describes an industrial heat exchanger in which steam is used to heat the liquid product. The requirement on the control system is to retain the product temperature at 55 degrees centigrade. To satisfy the need of the latter processes, the flow rate of the product regularly changes within 1.5 – 3.0 L/min. Fix the flow rate of the product at 2.1 L/min. The transfer function from the steam flow rate to the product temperature is obtained by carrying out step tests:

$$\boxed{G(s) = \frac{0.54e^{-15s}}{(15s+1)^2}}$$

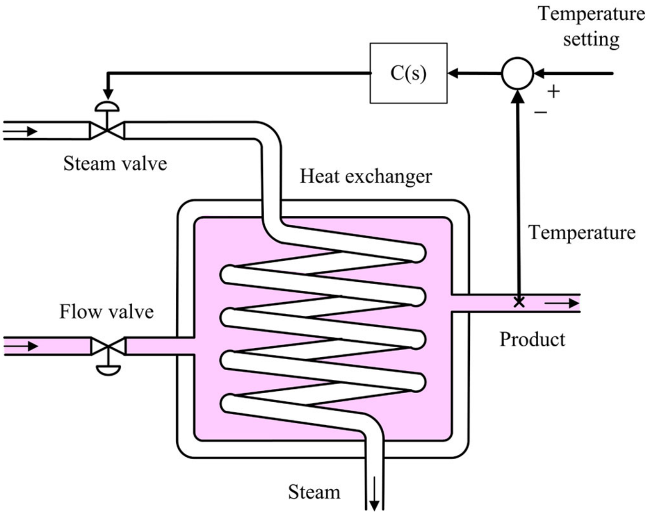

Figure: Industrial heat exchanger

Example (ctd.1)

The time delay depends on the flow rate of the product. When the flow rate changes in the prescribed range, the time delay changes within $10-20$ seconds

The design requirement is that the overshoot does not exceed 10% for the worst case. The controller of the second-order plant is obtained as follows:

$$\boxed{C(s)=\frac{1}{0.54}\frac{(15s+1)^2}{\lambda^2 s^2+(2\lambda+15)s}}$$

The parameter is taken as $\lambda=0.9\theta$. For the sake of comparison, a plant of reduced-order is computed:

$$G(s)=\frac{0.54e^{-21s}}{25s+1}$$

Example (ctd.2)

and $\lambda = 0.78\theta$ is taken for the controller of the first-order plant:

$$\boxed{C(s) = \frac{1}{0.54}\frac{(25s+1)(11.5s+1)}{\lambda^2 s^2 + (2\lambda + 11.5)s}.}$$

A unit step reference is added at $t = 0$ and a unit step load is added at $t = 200$. The nominal responses of the closed-loop system are shown in Figure. Since the two roots of the plant are real, the responses given by the two controllers are similar. Assume that the flow rate of the product decreases to the lowest so that the time delay becomes 20 seconds. Responses of this worst case are shown in Figure. The overshoot of the closed-loop system increases to 10% by the square

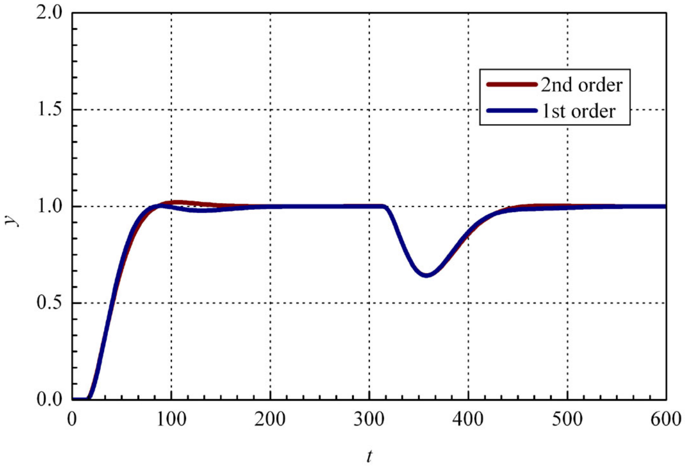

Figure: Responses of the nominal plant

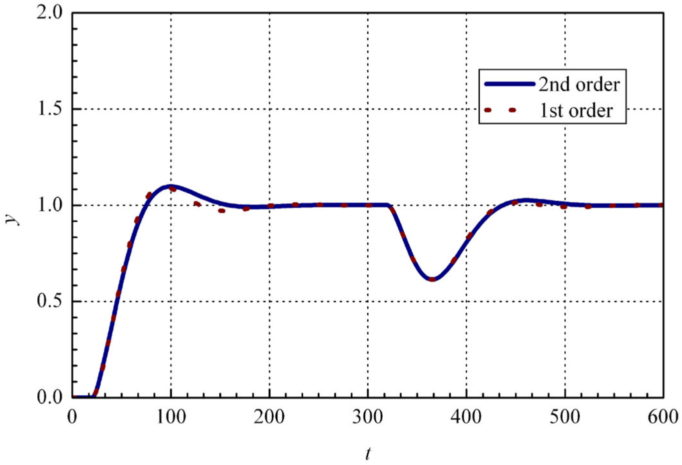

Figure: Responses of the worst case

4.6 All Stabilizing PID Controllers for Stable Plants

One might encounter such a case in practice: Even when the parameters of a PID controller are chosen in random, the closed-loop system still works well. Unfortunately, not every time one can find appropriate parameters, since it is not clear the range of the PID parameters for which the feedback system is stable

Goal of this section: Determine the set of controller parameters that guarantees the stability of the closed-loop system

The attention here is put on the first-order plant with time delay:

$$G(s) = \frac{Ke^{-\theta s}}{\tau s + 1}$$

To simplify matters, the standard PID controller is considered:

$$C(s) = K_C + \frac{K_I}{s} + K_D s$$

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$$C(s) = K_C + \frac{K_I}{s} + K_D s$$

Theorem

The plant can be stabilized by the PID controller if and only if the controller parameters satisfy

$$-\frac{1}{K} < K_C < K_T$$

where

$$K_T = \frac{1}{K}\left[\frac{\tau}{\theta}\alpha_1 \sin(\alpha_1) - \cos(\alpha_1)\right]$$

and α_1 is the solution of the equation

$$\tan(\alpha) = -\frac{\tau}{\tau+\theta}\alpha$$

in the interval $(0, \pi)$.

Theorem (ctd.1)

The complete stabilizing region is given by

1. *For* $K_C \in (-1/K, 1/K]$*, the stabilizing region of the integral constant and the derivative constant is the trapezoid in Figure.*
2. *For* $K_C \in (1/K, K_T)$*, the stabilizing region of the integral constant and the derivative constant is the quadrilateral in Figure.*

Here

$$
\begin{aligned}
z &= \theta\omega \\
m(z) &= \frac{\theta^2}{z^2} \\
b(z) &= -\frac{\theta}{Kz}\left[\sin(z) + \frac{\tau}{\theta} z\cos(z)\right] \\
w(z) &= \frac{z}{K\theta}\left\{\sin(z) + \frac{\tau}{\theta} z[\cos(z) + 1]\right\}
\end{aligned}
$$

Theorem (ctd.2)

and $z_j (j = 1, 2, \ldots)$ *are the positive real roots of*

$$KK_C + \cos(z) - \frac{\tau}{\theta} z \sin(z) = 0$$

These roots are arranged in increasing order of magnitude.

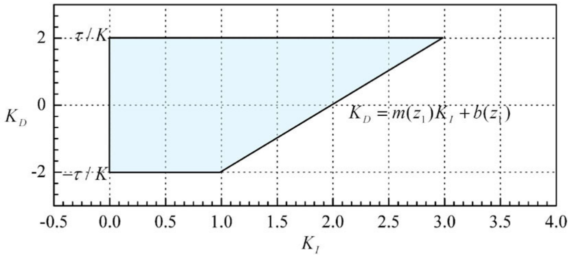

Figure: Stabilizing region for $K_C \in (-1/K, 1/K]$(From Silva et al., 2002. Reprinted by permission of the IEEE)

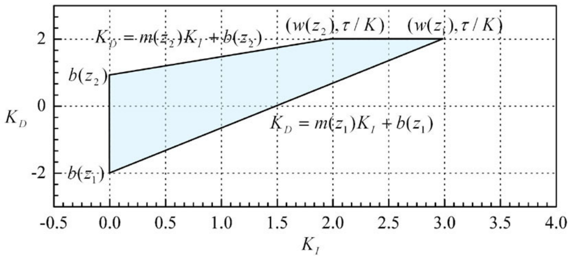

Figure: Stabilizing region for $K_C \in (1/K, K_T)$(From Silva et al., 2002. Reprinted by permission of the IEEE)

Proof.

The proof is only sketched. The characteristic polynomial of the system is in the form of a quasi-polynomial:

$$\delta(s) = K(K_I + K_C s + K_D s^2)e^{-\theta s} + (1 + \tau s)s$$

Proof ctd.1.

Since $e^{\theta s}$ does not have any finite zeros, the following quasi-polynomial is considered instead:

$$\delta^*(s) = K(K_I + K_C s + K_D s^2) + (1 + \tau s)se^{\theta s}$$

$\delta^*(s)$ and $\delta(s)$ are equivalent for stability analysis. Rewrite $\delta^*(s)$ as

$$\delta^*(j\omega) = \delta_r(\omega) + j\delta_i(\omega)$$

where $\delta_r(\omega)$ and $\delta_i(\omega)$ represent the real part and imaginary part of $\delta^*(j\omega)$, respectively.

$$\delta_r(\omega) = KK_I - KK_D\omega^2 - \omega\sin(\theta\omega) - \tau\omega^2\cos(\theta\omega)$$
$$\delta_i(\omega) = \omega[KK_C + \cos(\theta\omega) - \tau\omega\sin(\theta\omega)]$$

Proof ctd.2.

It can be seen that K_C only affects the imaginary part of $\delta^*(j\omega)$ whereas K_I and K_D only affect the real part of $\delta^*(j\omega)$. It can be proved that $\delta^*(s)$ is stable if and only if

1. $\delta_r(\omega)$ and $\delta_i(\omega)$ have only simple real roots and these roots interlace.
2. $E(\omega_0) := \delta_i'(\omega_0)\delta_r(\omega_0) - \delta_i(\omega_0)\delta_r'(\omega_0) > 0$ for some ω_0 in $(-\infty, +\infty)$.

In what follows it will be examined when the two conditions hold. First, check the second condition. Since $z = \theta\omega$, the real part and the imaginary part of $\delta^*(j\omega)$ can be, respectively, expressed as

$$\delta_r(z) = KK_I - \frac{KK_D}{\theta^2}z^2 - \frac{1}{\theta}z\sin(z) - \frac{\tau}{\theta^2}z^2\cos(z)$$
$$\delta_i(z) = \frac{z}{\theta}[KK_C + \cos(z) - \frac{\tau}{\theta}z\sin(z)]$$

Proof ctd.3.

Take $\omega_0 = z_0 = 0$. Then $\delta_r(z_0) = KK_I$ and $\delta_i(z_0) = 0$. On the other hand,

$$E(z_0) = \frac{KK_C + 1}{\theta} KK_I$$

If pick $K_I > 0, K_C > -1/K$ or $K_I < 0, K_C < -1/K$, then

$$E(z_0) > 0$$

Now check the first condition. Plotting the terms involved in the equation $\delta_i(z) = 0$ and graphically examining the nature of the solution, it can be obtained that the roots are all real if and only if $K_C \in (-1/K, K_T)$.
Furthermore, compute the roots of the imaginary part by letting $\delta_i(z) = 0$. Evidently, one root is $z_0 = 0$.

Proof ctd.4.

The other roots $z_j (j = 1, 2, ...)$ are given by the equation

$$KK_C + \cos(z) - \frac{\tau}{\theta} z \sin(z) = 0.$$

Arrange these roots in increasing order of magnitude. By evaluating $\delta_r(z)$ at $z_j (j = 0, 1, ...)$, it can be proved that K_I and K_D for the roots of $\delta_r(z)$ and $\delta_i(z)$ to interlace are determined by

$$\begin{aligned} K_I &> 0 \\ (-1)^j K_D &< (-1)^j m(z_j) K_I + (-1)^j b(z_j), \quad j = 1, 2, ... \end{aligned}$$

It is now shown that all these regions do have a nonempty intersection.

Proof ctd.5.

First, it is observed that the slopes $m(z_j)$ of the boundary lines of these regions decrease with z_j. The limit is

$$\lim_{j \to \infty} m(z_j) = 0$$

With this in mind, the following observations are obtained:With this in mind, the following observations are obtained:

1. When $K_C \in (-1/K, 1/K)$, the intersection is given by the trapezoid sketched in Figure. This is obtained by that
   1. $b(z_j) < b(z_{j+2}) < -\tau/K$ for odd values of j.
   2. $b(z_j) > \tau/K$ and $b(z_j) \to \tau/K$ as $j \to \infty$ for even values of j.
   3. $0 < v(z_j) < v(z_{j+2})$ for odd values of j, where

$$v(z) = \frac{z}{K\theta}\left\{\sin(z) + \frac{\tau}{\theta} z[\cos(z) - 1]\right\}.$$

Proof ctd.6.

2 When $K_C \in (-1/K, 1/K_T)$, the intersection is given by the quadrilateral sketched in Figure. This is obtained by that

1. $b(z_j) > b(z_{j+2}) > -\tau/K$ for odd values of j.
2. $b(z_j) < b(z_{j+2}) < \tau/K$ for even values of j.
3. $w(z_j) > w(z_{j+2}) > 0$ for even values of j.
4. $b(z_1) < b(z_2)$, $w(z_1) > w(z_2)$.

So far, the interlacing property, as well as that the roots of $\delta_i(z) = 0$ are all real for $K_C \in (1/K, K_T)$, has been proven. The two conditions can be used to prove that $\delta_r(z) = 0$ has only real roots.

Therefore, for $(1/K, K_T)$ there is a solution to the PID stabilization problem of the first-order plant with time delay. For those values of K_C outside this range, the PID stabilization problem does not have a solution.

$\square$

End of Chapter 4