Chapter 11 Classical Design Methods for MIMO Systems

11.1 Interaction Analysis

Feature of MIMO control systems: There exists interaction (or coupling) between inputs and outputs, that is, each input may affect more than one outputs. This feature makes the design of MIMO systems very challenging

The first step in many design methods for MIMO systems is to analyze the extent of interaction

Purpose: Determine proper pairings between plant inputs and plant outputs so that the plant output and the plant input that have the largest effect on each other are matched up

Purpose: An extensively adopted method for interaction analysis is Relative Gain Array (RGA). It provides a measure for the steady-state gain between a given input-output pairing. By selecting sensitive input-output connections, interaction can be reduced

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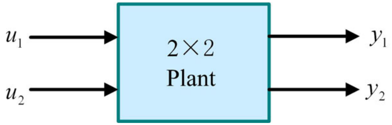
Consider the system with two inputs u_1 and u_2 and two outputs y_1 and y_2, where each output is affected by both inputs



The RGA is defined as

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix}$$

where

$$\lambda_{ij} = \frac{\text{Open-loop gain } k_{ij} \text{ between } y_i \text{ and } u_j}{\text{Closed-loop gain } a_{ij} \text{ between } y_i \text{ and } u_j}, i, j = 1, 2$$

is called the relative gain between y_i and u_j. It should be emphasized that the relative gain λ_{ij} is entirely unrelated to the performance degree, for which a similar symbol is used

The computation of RGA involves three steps:

1. Calculate open-loop gains.
2. Calculate closed-loop gains.
3. Calculate the RGA.

The open-loop gains can be expressed as

	u_1	u_2
y_1	$k_{11} = \left.\dfrac{\Delta y_1}{\Delta u_1}\right\vert_{u_2}$	$k_{12} = \left.\dfrac{\Delta y_1}{\Delta u_2}\right\vert_{u_1}$
y_2	$k_{21} = \left.\dfrac{\Delta y_2}{\Delta u_1}\right\vert_{u_2}$	$k_{22} = \left.\dfrac{\Delta y_2}{\Delta u_2}\right\vert_{u_1}$

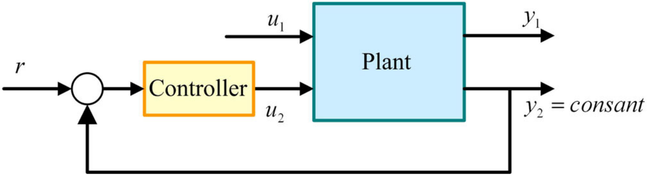
which are from the steady-state open-loop relationship between the inputs and outputs:

$$\begin{aligned} \Delta y_1 &= k_{11}\Delta u_1 + k_{12}\Delta u_2 \\ \Delta y_2 &= k_{21}\Delta u_1 + k_{22}\Delta u_2 \end{aligned}$$

The open-loop gain can be determined by utilizing experimental tests. To evaluate k_{11}, for example, one can make a small change Δu_1 in u_1 while the plant is operated at the steady state and u_2 is kept constant (that is, $\Delta u_2 = 0$). Let Δy_1 be the output offset. The open-loop gain between y_1 and u_1 is given by

$$k_{11} = \left.\frac{\Delta y_1}{\Delta u_1}\right|_{u_2}$$

Assume that instead of keeping u_2 constant, one makes a small change in u_1 and simultaneously manipulates u_2 to bring y_2 back to the value it had before the change in u_1 is made. This can be reached by a closed-loop system shown in Figure



The closed-loop gain between y_1 and u_1 is obtained as follows:

$$a_{11} = \left.\frac{\Delta y_1}{\Delta u_1}\right|_{y_2}$$

The gain a_{11} reflects how y_1 responds to a change in u_1 when y_2 is kept constant.

The relative gains λ_{ij} can be computed by utilizing the obtained k_{ij} and a_{ij}. As a_{ij} is not independent of k_{ij}, λ_{ij} can also be computed employing only k_{ij}. According to the definition of the closed-loop gain, $a_{11} = \Delta y_1/\Delta u_1$ when $\Delta y_2 = 0$. Then

$$0 = k_{21}\Delta u_1 + k_{22}\Delta u_2$$

Solving for Δu_2 results in

$$\Delta u_2 = -\frac{k_{21}}{k_{22}}\Delta u_1$$

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Solving for Δu_2 results in

$$\Delta u_2 = -\frac{k_{21}}{k_{22}}\Delta u_1$$

It follows that

$$\Delta y_1 = k_{11}\Delta u_1 - \frac{k_{12}k_{21}}{k_{22}}\Delta u_1$$

Therefore,

$$\begin{aligned} a_{11} &= \left.\frac{\Delta y_1}{\Delta u_1}\right|_{\Delta y_2=0} \\ &= k_{11} - \frac{k_{12}k_{21}}{k_{22}} \end{aligned}$$

The relative gain is

$$\begin{aligned} \lambda_{11} &= \frac{k_{11}}{a_{11}} \\ &= \frac{k_{11}k_{22}}{k_{11}k_{22} - k_{12}k_{21}} \end{aligned}$$

A useful property of the relative gain matrix is that each row and each column sums to 1. Thus, in a 2×2 system, only one of the four elements needs to be explicitly computed

The relative gain provides a useful measure for interaction. In particular:

1. If $\lambda_{ij} = 0$, the open-loop gain is zero. u_j does not have effect on y_i
2. If $\lambda_{ij} = 1$, the loop consisting of u_j and y_i is not affected by other loops
3. If $0 < \lambda_{ij} < 1$, there exists interaction among different loops. The worst case is $\lambda_{ij} = 0.5$
4. If $\lambda_{ij} < 0$, the open-loop gain is in the opposite direction to the closed-loop gain. This case should be avoided

The rule to reduce the interaction by pairing plant inputs and outputs is that the control loops should be selected in such a way that the relative gains are positive, and as close as possible to unity. Since only steady-state responses are considered, the rule does not guarantee the minimum dynamic interaction

The definition of relative gains and their use in selecting control loops are not limited to systems with two inputs and two outputs. The extension to $n \times n$ systems is straightforward. The relative gains can be computed with the following procedure for $n \times n$ systems

First, arrange the k_{ij}s in a matrix:

$$\mathbf{K} = \begin{bmatrix} k_{11} & \cdots & k_{1n} \\ \vdots & \ddots & \vdots \\ k_{n1} & \cdots & k_{nn} \end{bmatrix}$$

Then, compute a new matrix, by first inverting, then transposing the matrix $\mathbf{K}$:

$$(\mathbf{K}^{-1})^T = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{bmatrix}$$

The element in the ith row and jth column of $(\mathbf{K}^{-1})^T$ is the reciprocal of a_{ij}, that is,

$$c_{ij} = \frac{1}{a_{ij}}$$

The relative gain array is given by

$$\mathbf{\Lambda} = \mathbf{K} \otimes (\mathbf{K}^{-1})^T$$

where "$\otimes$" denotes the element-by-element product

Example

Blending is a frequently encountered process in industry. For example in a paper-making process, the thick pulp from the stock preparation system is blended with the recycled water, and then delivered to the head box

Example (ctd.1)

Choose the flow rate of thick pulp, u_1, and the flow rate of recycled water, u_2, as plant inputs. The system outputs are the flow rate of thin pulp, y_1, and its consistence, y_2. When the condenser paper is produced, the consistence of the thick pulp and the recycled water are 0.66% and 0.03%, respectively. The desired flow rate and consistence of thin pulp are 152 kg/min and 0.25%, respectively. The mass balance yields

$$\begin{aligned} 152 &= u_1 + u_2 \\ 152 \times 0.25\% &= u_1 \times 0.66\% + u_2 \times 0.03\% \end{aligned}$$

The steady-state solution for the flow rates of u_1 and u_2 is

$$u_1 = 53.08, u_2 = 98.92$$

Example (ctd.2)

Change u_1 by one unit (that is, u_1 changes from 53.08 to 54.08) while keep u_2 constant. The following steady-state outputs are obtained:

$$y_1 = 153, y_2 = 0.2527\%$$

Therefore,

$$k_{11} = \left.\frac{\Delta y_1}{\Delta u_1}\right|_{u_2} = \frac{1}{1} = 1$$

Change u_1 by one unit (that is, u_1 changes from 53.08 to 54.08) while keep y_2 the same. We have

$$y_1 = 154.87, u_2 = 100.79$$

Example (ctd.3)

Then

$$a_{11} = \left.\frac{\Delta y_1}{\Delta u_1}\right|_{y_2} = \frac{1.87}{1} = 1.87$$

Consequently, the relative gain array is

$$\mathbf{\Lambda} = \begin{bmatrix} 0.535 & 0.465 \\ 0.465 & 0.535 \end{bmatrix}$$

The two loops with minimum interaction are formed when y_1 is controlled by u_1 and y_2 is controlled by u_2

11.2 Decentralized Controller Design

Once proper loop pairings are determined, the next step is to design a controller for the MIMO system

Decentralized control: In decentralized control, multiple independent SISO controllers are designed. Each controller uses one plant input to control a preassigned output. Feedback is utilized to overcome the interaction

Such a system can always be arranged such that the controller is diagonal

Advantage of decentralized control:

1. The approach is easy to understand
2. Good control can be reached in many cases
3. The complexity and the cost of hardware are low

Compared to the system with a full controller matrix, the constraint imposed by the decentralized control on the controller structure leads to performance deterioration

Tradeoff in design: The designer must weigh which aspect is the most important, the performance or the simplicity

Assume that the plant is denoted by an $n \times n$ transfer function matrix $\mathbf{G}(s)$:

$$\mathbf{G}(s) = \begin{bmatrix} G_{11}(s) & \cdots & G_{1n}(s) \\ \vdots & \ddots & \vdots \\ G_{n1}(s) & \cdots & G_{nn}(s) \end{bmatrix}$$

and the controller is denoted by an $n \times n$ transfer function matrix $\mathbf{C}(s)$

Without loss of generality, in a decentralized control system $\mathbf{C}(s)$ is diagonal:

$$\mathbf{C}(s) = \mathrm{diag}\left\{C_{11}(s), \ldots, C_{nn}(s)\right\}$$

The reference $\mathbf{r}(s)$ is a vector of $n \times 1$ dimension:

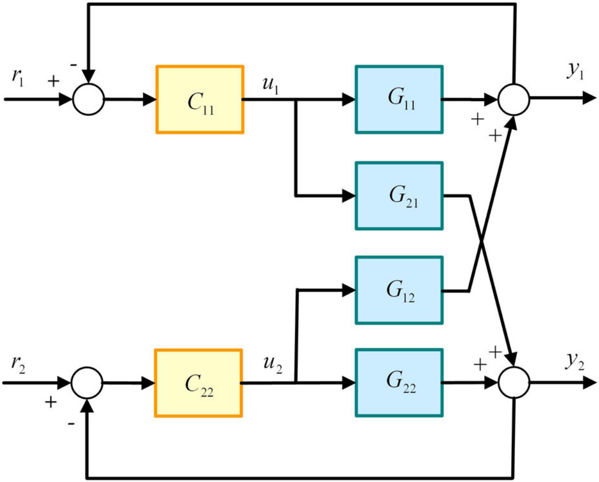
$$\mathbf{r}(s) = \begin{bmatrix} r_1(s) & r_2(s) & \ldots & r_n(s) \end{bmatrix}^T$$

and the system output $\mathbf{y}(s)$ is a vector of $n \times 1$ dimension as well:

$$\mathbf{y}(s) = \begin{bmatrix} y_1(s) & y_2(s) & \ldots & y_n(s) \end{bmatrix}^T$$

Then we have

$$\mathbf{y}(s) = \mathbf{G}(s)\mathbf{C}(s)[\mathbf{I} + \mathbf{G}(s)\mathbf{C}(s)]^{-1}\mathbf{r}(s)$$



The control structure for the simplest case (that is, the 2×2 case) is shown in Figure

The characteristic equation of the system is

$$\det[\mathbf{I} + \mathbf{G}(s)\mathbf{C}(s)] = 0$$

This equation can be used to test the closed-loop stability. For a stable plant, if the Nyquist plot of $\det[\mathbf{I} + \mathbf{G}(s)\mathbf{C}(s)]$ encircles the origin, the closed-loop system is unstable. Define a scalar function $W_c(s)$:

$$W_c(s) = -1 + \det\left[\mathbf{I} + \mathbf{G}(s)\mathbf{C}(s)\right]$$

The closer the Nyquist plot of $W_c(s)$ to the point (-1, 0), the closer the closed-loop system to instability. Define the closed-loop logarithm modulus for MIMO systems as follows:

$$L_c = 20\lg\left|\frac{W_c(j\omega)}{1 + W_c(j\omega)}\right|$$

By testing on a large number of MIMO plants, it is found that the following choice can provide reasonable tradeoff between stability and performance:

$$\max(L_c) = 2n$$

In a SISO system, $\max(L_c)$ is the resonance peak.

Different controller parameters may reach $\max(L_c) = 2n$ for the same plant. When better response is desired, further tuning has to be carried out

The SISO H_∞ or H_2 design methods introduced in foregoing chapters can be used to design a decentralized controller. The design procedure is as follows:

1. Calculate SISO controllers for each individual loop.
2. Close all loops. Take the same performance degrees temporarily: $\lambda_1 = \lambda_2 = ... = \lambda_n$. Increase the performance degrees from small to large so that the closed-loop system is stable.
3. Tune each performance degree to reach the required closed-loop response (for example, $\max(L_c) = 2n$).

Advantage of the decentralized $\mathbf{H}_\infty$ or $\mathbf{H}_2$ design: Each channel can be tuned easily for the required response; the tuning does not require exact model about the uncertainty

Analysis of the stability:

For the original system,

$$\begin{aligned}\mathbf{T}(s) &= \mathbf{G}(s)\mathbf{C}(s)[\mathbf{I} + \mathbf{G}(s)\mathbf{C}(s)]^{-1} \\ \mathbf{S}(s) &= [\mathbf{I} + \mathbf{G}(s)\mathbf{C}(s)]^{-1}\end{aligned}$$

In decentralized control:
Nominal plan—Diagonal elements are regarded as a
Uncertainty—Nondiagonal elements

Write the nominal plant as follows:

$$\mathbf{G_a}(s) = \mathrm{diag}\{G_{11}(s), G_{22}(s), ..., G_{nn}(s)\}$$

The nominal plant and the controller constitute a new system:

$$\begin{aligned}\mathbf{T_a}(s) &= \mathbf{G_a}(s)\mathbf{C}(s)[\mathbf{I} + \mathbf{G_a}(s)\mathbf{C}(s)]^{-1} \\ \mathbf{S_a}(s) &= [\mathbf{I} + \mathbf{G_a}(s)\mathbf{C}(s)]^{-1}\end{aligned}$$

Regard the new system as the nominal one and the original system as the uncertain one. The uncertainty is described by the multiplicative output uncertainty $\delta_{\mathbf{a}}(s)$. Assume that $\mathbf{G}(s)$ and $\mathbf{G_a}(s)$ have the same RHP poles

Theorem

Assume that $\mathbf{T_a}(s)$ is stable. The closed-loop system $\mathbf{T}(s)$ is stable if and only if the Nyquist plot of $\det[\mathbf{I} + \delta_{\mathbf{a}}(s)\mathbf{T_a}(s)]$ does not encircle the origin.

Proof.

For the original system, we have the following identity:

$$\mathbf{I} + \mathbf{G}(s)\mathbf{C}(s) = [\mathbf{I} + \delta_{\mathbf{a}}(s)\mathbf{T_a}(s)][\mathbf{I} + \mathbf{G_a}(s)\mathbf{C}(s)]$$

Let the number of the unstable poles of $\mathbf{G}(s)$ and $\mathbf{G_a}(s)$ be k. $\mathbf{T}(s)$ is stable if and only if

$$\det[\mathbf{I} + \mathbf{G}(s)\mathbf{C}(s)] = \det[\mathbf{I} + \mathbf{G_a}(s)\mathbf{C}(s)]\det[\mathbf{I} + \delta_{\mathbf{a}}(s)\mathbf{T_a}(s)]$$

encircles the origin k times counterclockwise

Proof ctd.1.

Because $\mathbf{T_a}(s)$ is stable by assumption, $\det[\mathbf{I} + \mathbf{G_a}(s)\mathbf{C}(s)]$ encircles the origin k times counterclockwise. It is immediately known that $\det[\mathbf{I} + \delta_{\mathbf{a}}(s)\mathbf{T_a}(s)]$ should not encircle the origin □

Example

Basis weight and moisture content are the two primary controlled variables in paper-making processes. In general, the basis weight is controlled by adjusting the flow rate of stock. The larger the flow rate, the larger the basis weight. The moisture content can be controlled by adjusting the steam pressure of dryers. The larger the steam pressure, the higher the temperature of dryers and the less the moisture content

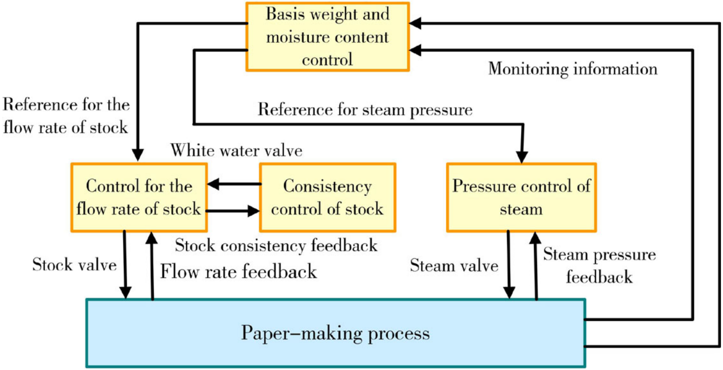


Figure: Control strategy of paper machines

Example (ctd.1)

For producing the paper of 78g/m^2, the model of a paper machine is

$$\boxed{\mathbf{G}(s) = \begin{bmatrix} \frac{5.15e^{-2.8s}}{1.8s+1} & \frac{-0.20e^{-1.2s}}{2.23s+1} \\ \frac{0.44e^{-2.8s}}{1.8s+1} & \frac{-1.26e^{-1.2s}}{2.23s+1} \end{bmatrix}}$$

Suppose that the H_∞ PID controller given by (**??**) is used:

$$\boxed{\mathbf{C}(s) = \begin{bmatrix} C_{11}(s) & 0 \\ 0 & C_{22}(s) \end{bmatrix}}$$

where

Example (ctd.2)

$$C_{11}(s) = \frac{1}{5.15} \frac{(1.8s+1)(1+1.4s)}{\lambda^2 s^2 + (2\lambda + 1.4)s}$$

$$C_{22}(s) = \frac{1}{1.26} \frac{(2.23s+1)(1+0.6s)}{\lambda^2 s^2 + (2\lambda + 0.6)s}$$

The controller parameters are taken as $\lambda_1 = 0.9\theta_{11}$, $\lambda_2 = 0.6\theta_{22}$, where θ_{11} and θ_{22} are time delays of the diagonal elements respectively. The response of the closed-loop system is fast and steady (Figure)

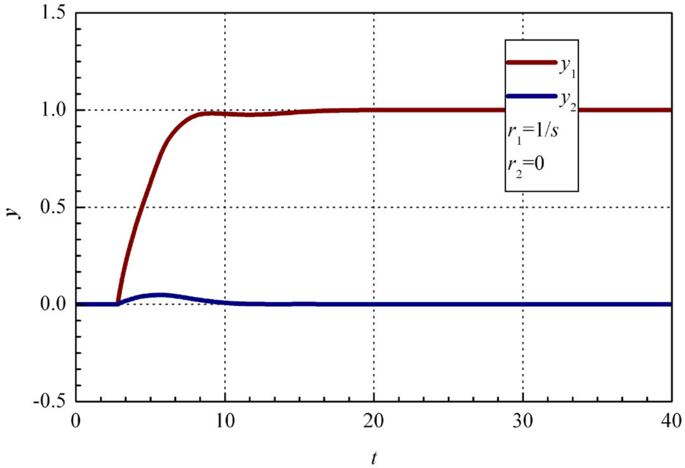


Figure: Responses of the decentralized H_∞ controller-1

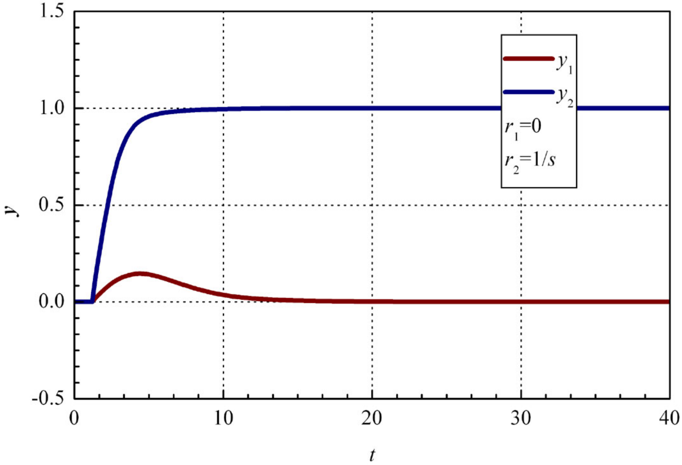


Figure: Responses of the decentralized H_∞ controller-2

11.3 Decoupler Design

When the interaction among control loops is weak, the decentralized control method works well

For the control loops with severe interaction, decoupling control is a better method

Decoupling control: In decoupling control, an additional compensation structure called decoupler is introduced to reduce the interaction

Function of a decoupler: Decompose a MIMO plant into a series of independent SISO plants. Then the SISO design methods (for example, the H_∞ method or H_2 method introduced in foregoing chapters) can be utilized to design MIMO control systems

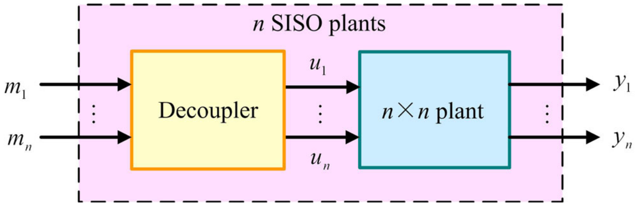
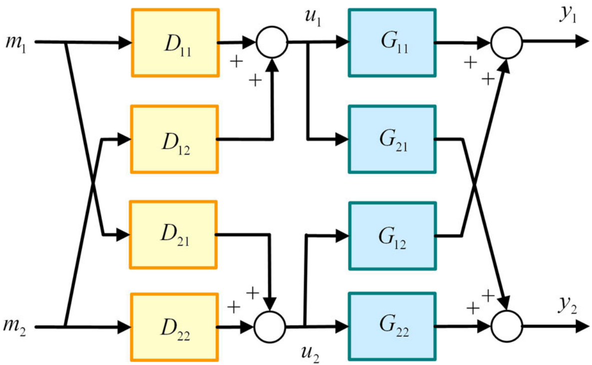


Figure shows the general decoupling structure



For ease of understanding, consider a 2×2 MP plant first. The decoupling system is shown in Figure, where the decoupler inputs are two new manipulated variables $m_1(s)$ and $m_2(s)$, and its outputs are the original manipulated variables $u_1(s)$ and $u_2(s)$

For the original plant we have

$$\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix}$$

The decoupler equation can be written as

$$\begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix} = \begin{bmatrix} D_{11}(s) & D_{12}(s) \\ D_{21}(s) & D_{22}(s) \end{bmatrix} \begin{bmatrix} m_1(s) \\ m_2(s) \end{bmatrix}$$

Then the equations for the plant-decoupler combination are

$$\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \begin{bmatrix} D_{11}(s) & D_{12}(s) \\ D_{21}(s) & D_{22}(s) \end{bmatrix} \begin{bmatrix} m_1(s) \\ m_2(s) \end{bmatrix}$$

The decoupler should be designed so that the non-diagonal elements are zero. In this case, $y_1(s)$ is only affected by $u_1(s)$ and $y_2(s)$ is only affected by $u_2(s)$

Let

$$\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} P_{11}(s) & 0 \\ 0 & P_{22}(s) \end{bmatrix} \begin{bmatrix} m_1(s) \\ m_2(s) \end{bmatrix}$$

We have

$$\begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \begin{bmatrix} D_{11}(s) & D_{12}(s) \\ D_{21}(s) & D_{22}(s) \end{bmatrix} = \begin{bmatrix} P_{11}(s) & 0 \\ 0 & P_{22}(s) \end{bmatrix}$$

This equality involves 4 equations and 6 unknowns. The solution is not unique. For example, the following are possible solutions:

$$D_{11}(s) = 1, D_{12}(s) = -\frac{G_{12}(s)}{G_{11}(s)}, D_{21}(s) = -\frac{G_{21}(s)}{G_{22}(s)}, D_{22}(s) = 1$$

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$$D_{11}(s) = -\frac{G_{22}(s)}{G_{21}(s)}, D_{12}(s) = 1, D_{21}(s) = 1, D_{22}(s) = -\frac{G_{11}(s)}{G_{12}(s)}$$

For the obtained solution, $P_{11}(s)$ and $P_{22}(s)$ can be determined as follows:

$$P_{11}(s) = G_{11}(s)D_{11}(s) + G_{12}(s)D_{21}(s)$$
$$P_{22}(s) = G_{21}(s)D_{12}(s) + G_{22}(s)D_{22}(s)$$

To obtain an unique solution, three typical methods were proposed in literature. The first is to take the inverse of the plant as the decoupler, that is,

$$\begin{bmatrix} P_{11}(s) & 0 \\ 0 & P_{22}(s) \end{bmatrix} = \mathbf{I}$$

The second is to take 1 as the diagonal elements of the decoupler:

$$\begin{bmatrix} D_{11}(s) & D_{12}(s) \\ D_{21}(s) & D_{22}(s) \end{bmatrix} = \begin{bmatrix} 1 & D_{12}(s) \\ D_{21}(s) & 1 \end{bmatrix}$$

And the third is to take 0 as the diagonal elements of the decoupler:

$$\begin{bmatrix} D_{11}(s) & D_{12}(s) \\ D_{21}(s) & D_{22}(s) \end{bmatrix} = \begin{bmatrix} 0 & D_{12}(s) \\ D_{21}(s) & 0 \end{bmatrix}$$

Question: Which decoupling scheme should the designer choose among the three?

Two aspects should be considered for the question:

- The first is whether the obtained decoupler is realizable. Some methods cannot give a decoupler that is physically realizable, in particular, when the plant involves RHP zeros or time delays
- The second is whether a simple decoupler can be obtained. The ultimate objective of the decoupler design is to eliminate the effect of interactions. If the decoupler is realizable, the simpler is certainly the better

And the third is to take 0 as the diagonal elements of the decoupler:

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Now consider the decoupler design problem for a general plant. The design equations can be conveniently summarized by using matrix notations

Let the plant be an $n \times n$ transfer function matrix $\mathbf{G}(s)$ and the decoupler be an $n \times n$ transfer function matrix $\mathbf{D}(s)$. We have

$$\mathbf{y}(s) = \mathbf{G}(s)\mathbf{D}(s)\mathbf{m}(s)$$

Assume that the decoupled plant is an $n \times n$ diagonal matrix $\mathbf{P}(s)$. It is desirable that

$$\mathbf{y}(s) = \mathbf{P}(s)\mathbf{m}(s)$$

By comparison, the decoupler is given by

$$\mathbf{D}(s) = \mathbf{G}(s)^{-1}\mathbf{P}(s)$$

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Mathematically, the calculation of the decoupler is trivial. If $\mathbf{P}(s)$ is given (for example, it is taken to be a unity matrix), $\mathbf{D}(s)$ can be determined uniquely as long as $\mathbf{G}(s)$ is not singular

Non-singularity of a transfer function matrix: Means that its determinant is not identically zero, or equivalently that the transfer function matrix is not singular for every s in the set of complex numbers, except for a finite number of points. The inverse of a non-singular $\mathbf{G}(s)$ can be expressed as

$$\mathbf{G}^{-1}(s) = \frac{\mathrm{adj}[\mathbf{G}(s)]}{\det[\mathbf{G}(s)]}$$

where $\mathrm{adj}(\cdot)$ denotes the adjoint. The determinant of $\mathbf{G}(s)$ can be calculated as a signed sum of the permutations taking one and only one element from every row and column of $\mathbf{G}(s)$:

$$\det[\mathbf{G}(s)] = \sum_{j_1 j_2 \ldots j_n} [\mathrm{sgn}(j_1 j_2 \ldots j_n)] G_{1j_1} G_{2j_2} \ldots G_{nj_n}$$

where $j_1 j_2 \ldots j_n$ denotes a permutation of the number $1, 2, \ldots, n$. The value of $\mathrm{sgn}(j_1 j_2 \ldots j_n)$ can be found as follows:

$$\mathrm{sgn}(j_1 j_2 \ldots j_n) = \left\{ \begin{array}{ll} 1 & \text{if the number of permutation inversion is even} \\ -1 & \text{if the number of permutation inversion is odd} \end{array} \right\}$$

The element of the adjoint matrix, $\mathrm{adj}[\mathbf{G}(s)]$, is the cofactor of $G_{ij}(s)$, which is the signed determinant of $\mathbf{G}(s)$ with row i and the column j removed

Nevertheless, the problem is not so easy from a control theory point of view:

- The obtained decoupler may not be physically realizable
- The decoupled system may be internally unstable when the plant is NMP or unstable

These problems can be solved well using the methods in the next two chapters where $j_1 j_2 ... j_n$ denotes a permutation of the number $1, 2, ..., n$. The value of $\mathrm{sgn}(j_1 j_2 ... j_n)$ can be found as follows:

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End of Chapter 11