Chapter 8 Control of Unstable Plants

8.1 Controller Parameterization for General Plants

Three types of plants: Stable, integrating, and unstable

Why categorizing: Controllers can be designed aiming at the reduced scope of plants, so that the design is more effective and simple controllers are easier to obtain

Plants considered in this section: Unstable plants with time delay

Difficulties in the design:

1. The existence of RHP poles makes the stabilization of the closed-loop system difficult to achieve
2. The combined effect of the RHP poles and the time delay greatly limits the achievable performance

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Assumption

For the control system with an unstable plant, there exists a limit on the ratio of the time constant to the time delay. If no further explanation is given, it is assumed that the condition is satisfied

The design method in this chapter is based on a new parameterization. As a matter of fact, special cases of the new parameterization were already used in the foregoing chapters

Why not the Youla parameterization?

1. It cannot be directly used for a plant with time delay
2. To obtain it, one has to compute the coprime factorization of the plant. No analytical methods are available
3. The $Q(s)$ in the general parameterization no longer corresponds to the IMC controller

Consider the unity feedback loop, in which the transfer function of the plant is given by

$$G(s) = \frac{KN_+(s)N_-(s)}{M_+(s)M_-(s)} e^{-\theta s}$$

Assume that $G(s)$ has r_p unstable poles and the unstable pole p_j is of l_j multiplicity ($j = 1, 2, ..., r_p$); that is,

$$M_+(s) = \prod_{j=1}^{r_p} (s - p_j)^{l_j}$$

Define

$$Q(s) = \frac{C(s)}{1 + G(s)C(s)}$$

which corresponds to the IMC controller

The closed-loop system is internally stable, if and only if all elements in the transfer matrix $\mathbf{H}(s)$ are stable:

$$\mathbf{H}(s) = \begin{bmatrix} G(s)Q(s) & G(s)[1 - G(s)Q(s)] \\ Q(s) & -G(s)Q(s) \end{bmatrix}$$

Theorem

The unity feedback system with a general plant $G(s)$ is internally stable if and only if

1. *$Q(s)$ is stable,*
2. *$[1 - G(s)Q(s)]G(s)$ is stable.*

Or equivalently,

1. *$Q(s)$ is stable,*
2. *$1 - G(s)Q(s)$ has zeros wherever $G(s)$ has unstable poles,*
3. *All RHP zero-pole cancellations in $[1 - G(s)Q(s)]G(s)$ are removed.*

Example

This example is used to illustrate that the third condition is necessary.

Consider the plant with the transfer function

$$G(s) = \frac{1}{s-1}$$

$G(s)$ has one simple RHP pole at $s = 1$. Construct a controller

$$C(s) = \frac{s-1}{e^{0.1s}(e^{-0.1}s - 0.1s + 0.1) - 1}$$

$Q(s)$ corresponding with this $C(s)$ is

$$Q(s) = \frac{s-1}{e^{0.1s}(e^{-0.1}s - 0.1s + 0.1)}$$

Example (ctd.1)

$Q(s)$ is stable. The first condition is satisfied. Furthermore,

$$1 - G(s)Q(s) = \frac{e^{0.1s}(e^{-0.1}s - 0.1s + 0.1) - 1}{e^{0.1s}(e^{-0.1}s - 0.1s + 0.1)}$$

It has zeros where $G(s)$ has unstable poles. The second condition is also satisfied.
However, the closed-loop system is internally unstable, because there exists a RHP zero-pole cancellation in $[1 - G(s)Q(s)]G(s)$, which cannot be removed

Remark 1: The case associated with the third condition occurs **only** in the system where the plant or the controller contains a time delay. If both the plant and the controller are rational, it is not necessary to consider the third condition

Remark 2: In control system design, $G(s)Q(s)$ is always stable. Since $[1 - G(s)Q(s)]G(s) = C^{-1}(s)Q(s)G(s)$, the third condition can be achieved by removing the RHP zero-pole cancellation in $C(s)$ through rational approximations

Theorem

All controllers that make the unity feedback control system internally stable can be parameterized as

$$C(s) = \frac{Q(s)}{1 - G(s)Q(s)}$$

where

$$Q(s) = \frac{Q_1(s)M_+(s)}{K}$$

Theorem (ctd.1)

$Q_1(s)$ is any stable transfer function that makes $Q(s)$ proper and satisfies

$$\lim_{s\to p_j}\frac{d^k}{ds^k}\left[1-\frac{Q_1(s)N_+(s)N_-(s)e^{-\theta s}}{M_-(s)}\right]=0,k=0,1,...,l_j-1$$

and all RHP zero-pole cancellations in $[1-G(s)Q(s)]G(s)$ are removed

Proof.

To guarantee the internal stability of the closed-loop system, first, $Q(s)$ should be stable. This implies that $Q(s)$ should be proper and $Q_1(s)$ should be stable.
Second, $[1-G(s)Q(s)]G(s)$ should be stable. This condition has three implications:

Proof ctd.1.

$Q(s)$ must cancel all RHP poles of $G(s)$, $1 - G(s)Q(s)$ must cancel all RHP poles of $G(s)$, and all RHP zero-pole cancellations in $[1 - G(s)Q(s)]G(s)$ are removed. All stable transfer functions that have zeros wherever $G(s)$ has RHP poles can be expressed as

$$Q(s) = \frac{Q_1(s)M_+(s)}{K}$$

where $Q_1(s)$ is a stable transfer function that makes $Q(s)$ proper. It follows that

$$1 - G(s)Q(s) = 1 - \frac{Q_1(s)N_+(s)N_-(s)e^{-\theta s}}{M_-(s)}$$

Proof ctd.2.

That $1 - G(s)Q(s)$ has zeros wherever $G(s)$ has RHP poles is equivalent to

$$\lim_{s \to s_j} \frac{d^k}{ds^k} \left[1 - \frac{Q_1(s)N_+(s)N_-(s)e^{-\theta s}}{M_-(s)} \right] = 0, k = 0, 1, ..., l_j - 1$$

$\square$

Corollary

Assume that $G(s)$ is a stable plant. That is, $M_+(s) = 1$. All controllers that make the unity feedback control system internally stable can be parameterized as

$$C(s) = \frac{Q(s)}{1 - G(s)Q(s)}$$

where $Q(s)$ is any stable transfer function.

Example

Consider a plant with the transfer function

$$G(s) = \frac{s-2}{(s-1)(s+2)}$$

The plant has only one simple unstable pole at $s = 1$. Then

$$Q(s) = (s-1)Q_1(s)$$

where $Q_1(s)$ is a stable transfer function satisfying

$$\lim_{s \to 1} \left[1 - Q_1(s) \frac{s-2}{s+2} \right] = 0$$

This is equivalent to

$$Q_1(s) = -3 + (s-1)Q_2(s)$$

Example (ctd.1)

where $Q_2(s)$ is any stable transfer function that makes $Q(s)$ proper. All controllers that make the unity feedback system internally stable can be parameterized as

$$C(s) = \frac{(s-1)(s+2)[-3+(s-1)Q_2(s)]}{(s+2)-(s-2)[-3+(s-1)Q_2(s)]}$$

Example

Consider the stabilizing problem of the plant

$$G(s) = \frac{1}{(s-1)(s-2)}$$

which has unstable poles at $s=1$ and $s=2$, respectively. Then

$$Q(s) = (s-1)(s-2)Q_1(s)$$

Example (ctd.1)

where $Q_1(s)$ is a stable transfer function satisfying

$$\lim_{s\to 1}[1 - Q_1(s)] = 0, \lim_{s\to 2}[1 - Q_1(s)] = 0$$

This is equivalent to

$$Q_1(s) = 1 + (s-1)(s-2)Q_2(s)$$

where $Q_2(s)$ is any stable transfer function that makes $Q(s)$ proper. All controllers that make the unity feedback system internally stable can be parameterized as

$$\begin{aligned} C(s) &= \frac{1 + (s-1)(s-2)Q_2(s)}{-Q_2(s)} \\ &= (s-1)(s-2) - \frac{-1}{Q_2(s)} \end{aligned}$$

When the system performance is considered, it is always desirable that the system has asymptotic tracking property. The parameterization can be further developed to cover the requirement on asymptotic tracking

Theorem

All controllers that make the unity feedback control system internally stable and have asymptotic tracking property for a step input can be parameterized as

$$C(s) = \frac{Q(s)}{1 - G(s)Q(s)}$$

where

$$Q(s) = \frac{[1 + sQ_2(s)]M_+(s)}{K}$$

Theorem (ctd.1)

$Q_2(s)$ is any stable transfer function that makes $Q(s)$ proper and satisfies

$$\lim_{s \to s_j} \frac{d^k}{ds^k}\left\{1 - \frac{[1 + sQ_2(s)]N_+(s)N_-(s)e^{-\theta s}}{M_-(s)}\right\} = 0, k = 0, 1, ..., l_j - 1$$

and all RHP zero-pole cancellations in $[1 - G(s)Q(s)]G(s)$ are removed

Proof.

$Q(s)$ should be stable and has zeros wherever $G(s)$ has RHP poles. Such a transfer function can be expressed as

$$Q(s) = \frac{Q_1(s)M_+(s)}{K}$$

proof ctd.1.

where $Q_1(s)$ is stable. If

$$\lim_{s\to 0}[1 - G(s)Q(s)] = 0$$

the closed-loop system possesses the asymptotic tracking property, which implies that

$$Q_1(s) = 1 + sQ_2(s)$$

where $Q_2(s)$ is a stable transfer function that makes $Q(s)$ proper. This leads to

$$1 - G(s)Q(s) = 1 - \frac{[1 + sQ_2(s)]N_+(s)N_-(s)e^{-\theta s}}{M_-(s)}$$

proof ctd.2.

Then $Q_2(s)$ should satisfy

$$\lim_{s \to s_j} \frac{d^k}{ds^k} \left\{ 1 - \frac{[1 + sQ_2(s)] N_+(s) N_-(s) e^{-\theta s}}{M_-(s)} \right\} = 0$$
$$k = 0, 1, ..., l_j - 1$$

□

Feature of the new parameterization: No coprime factorization is used. Instead, the properness of $Q(s)$ and the related constraints must be tested

In the design framework of this book, the parameterization is **only** used to derive the analytical design formula. It is not necessary to compute the parameterization

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8.2 H_∞ PID Controllers for Unstable Plants

Design for the First-Order Plant

Assume that the transfer function of the plant is

$$G(s) = \frac{K}{\tau s - 1} e^{-\theta s}$$

With the help of the first-order Taylor series expansion, the approximate plant is obtained as follows:

$$G(s) \approx \frac{K(1 - \theta s)}{\tau s - 1}$$

The plant has a RHP pole at $s = 1/\tau$. Obviously, θ and τ cannot be equal; otherwise, there would be a RHP zero-pole cancellation in the model. If the closed-loop system is internally stable, then

$$Q(s) = \frac{(\tau s - 1) Q_1(s)}{K}$$

To guarantee that the controller is physically realizable, a filter must be introduced: $Q(s) = Q_{opt}(s)J(s)$. The closed-loop system with the filter should be internally stable:

$$\lim_{s \to 1/\tau} [1 - G(s)Q(s)] = 0$$

and possesses the asymptotic tracking property:

$$\lim_{s \to 0} [1 - G(s)Q(s)] = 0$$

It is evident that a first-order filter cannot satisfy the requirements. Similar to that for the control of integrating plants, take

$$J(s) = \frac{\beta s + 1}{(\lambda s + 1)^2}$$

where λ is the performance degree and β is a positive real number. β is introduced to satisfy the above constraint

Elementary computation gives

$$\beta = \frac{\lambda^2 + 2\lambda\tau + \theta\tau}{\tau - \theta}$$

One readily obtain the suboptimal controller:

$$Q(s) = \frac{(\tau s - 1)(\beta s + 1)}{K(\lambda s + 1)^2}$$

It follows that

$$C(s) = \frac{Q(s)}{1 - G(s)Q(s)} = \frac{\alpha}{K}\left(1 + \frac{1}{\beta s}\right)$$

This is a PI controller. Here

$$\alpha = \frac{\lambda^2 + 2\lambda\tau + \theta\tau}{(\lambda + \theta)^2}$$

The closed-loop response can be quantitatively tuned by means of the performance degree. Nevertheless, the response is affected by the time constant in addition to the time delay

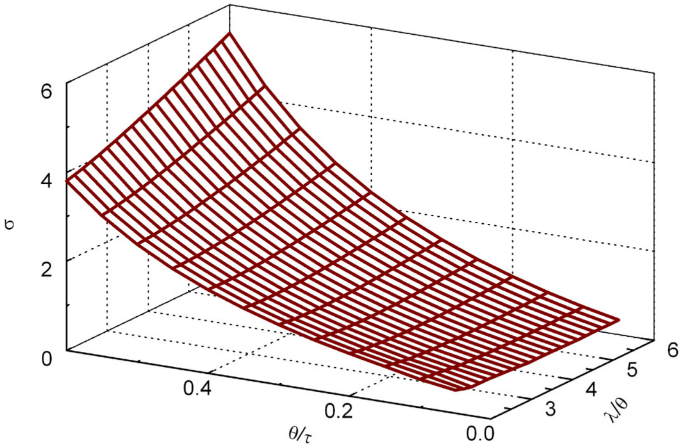

Figure: Overshoot of the H_∞ control system with an unstable plant

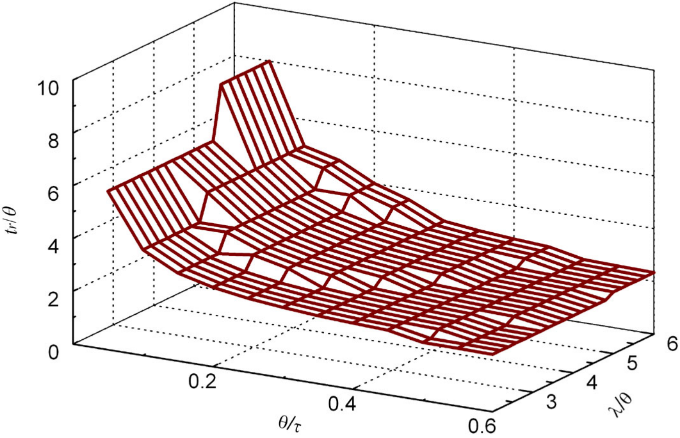

Figure: Rise time of the H$_\infty$ control system with an unstable plant

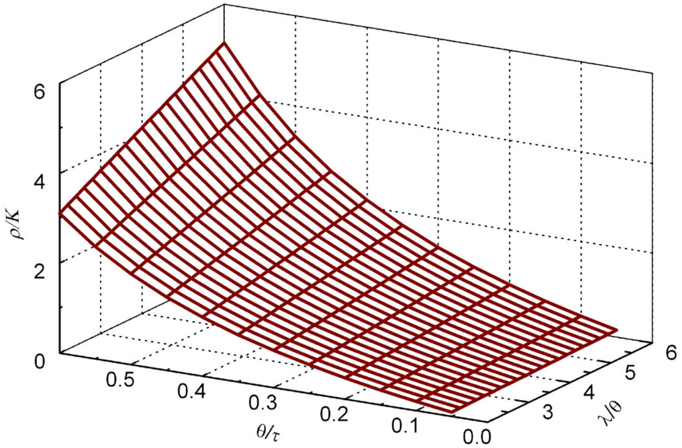

Figure: Perturbation peak of the H_∞ control system with an unstable plant

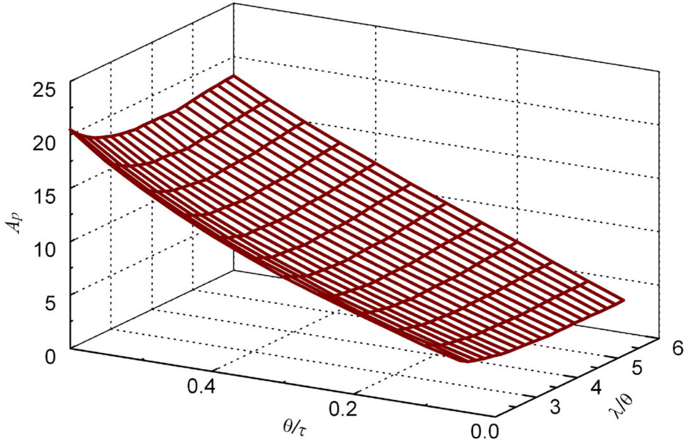

Figure: Resonance peak of the H_∞ control system with an unstable plant

Remark 1

To get a controller that makes the closed-loop system have desired properties, the requirement on $Q(s)$ for internal stability and asymptotic tracking is temporarily relaxed when designing the optimal controller $Q_{opt}(s)$. These requirements are satisfied by introducing a filter $J(s)$ to $Q_{opt}(s)$

Remark 2

It is possible to consider these requirements in designing $Q_{opt}(s)$. However, the obtained $Q_{opt}(s)$ is complicated. The mathematically precise reader can interpret the design in such a way: $Q(s)$ is directly obtained as the solution of the optimization problem

Design for the Second-Order Plant

One may use the second-order model with time delay:

$$G(s) = \frac{K}{(\tau_1 s - 1)(\tau_2 s + 1)} e^{-\theta s}$$

where τ_1 and τ_2 are two time constants. With the same design procedure, the following optimal controller can be obtained:

$$Q_{opt}(s) = \frac{(\tau_1 s - 1)(\tau_2 s + 1)}{K}$$

A filter that can guarantee the internal stability and the asymptotic tracking property is

$$J(s) = \frac{\beta s + 1}{(\lambda s + 1)^3}$$

where

$$\beta = \frac{\lambda^3 + 3\lambda^2 \tau_1 + 3\lambda \tau_1^2 + \theta \tau_1^2}{\tau_1(\tau_1 - \theta)}$$

A little algebra yields

$$C(s) = \frac{1}{K}\frac{(\tau_2 s + 1)(\beta s + 1)}{s(\lambda^3 s/\tau_1 + \alpha)}$$

where

$$\alpha = \frac{\lambda^3 + 3\lambda^2\tau_1 + 3\lambda\tau_1\theta + \theta^2\tau_1}{\tau_1(\tau_1 - \theta)}$$

If the PID controller is in the form of

$$C(s) = K_C\left(1 + \frac{1}{T_I s} + T_D s\right)\frac{1}{T_F s + 1}$$

controller parameters are

$$T_F = \frac{\lambda^3}{\tau_1\alpha} \qquad , T_I = \tau_2 + \beta$$

$$T_D = \frac{\tau_2\beta}{\tau_2 + \beta} \qquad , K_C = \frac{\tau_2 + \beta}{K\alpha}$$

8.3 H_2 PID Controllers for Unstable Plants

Design for the First-Order Plant

Design procedure for H_2 PID:

1. Parameterize all stabilizing controllers
2. Derive the optimal PID controller

The first-order Taylor series expansion is used to obtain the following plant:

$$G(s) \approx \frac{K(1-\theta s)}{\tau s - 1}$$

The controller that makes the closed-loop system internally stable and has the asymptotic tracking property can be expressed as

$$Q(s) = \frac{(\tau s - 1)[1 + sQ_1(s)]}{K}$$

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$$Q(s) = \frac{(\tau s - 1)[1 + sQ_1(s)]}{K}$$

$G(s)$ has a pole in the RHP. $Q_1(s)$ should satisfy

$$\lim_{s\to 1/\tau}[1 - G(s)Q(s)] = 0$$

for internal stability The performance index is taken as $\min \|W(s)S(s)\|_2$ and the weighting function is taken as $W(s) = 1/s$. Therefore,

$$\begin{aligned}
\|W(s)S(s)\|_2^2 &= \|W(s)[1 - G(s)Q(s)]\|_2^2 \\
&= \left\|\frac{1}{s} - \frac{1-\theta s}{s}[1 + sQ_1(s)]\right\|_2^2 \\
&= \|\theta - (1-\theta s)Q_1(s)\|_2^2 \\
&= \left\|\frac{\theta(1+\theta s)}{1-\theta s} - (1+\theta s)Q_1(s)\right\|_2^2 \\
&= \left\|\frac{2\theta}{1-\theta s}\right\|_2^2 + \|\theta + (1+\theta s)Q_1(s)\|_2^2
\end{aligned}$$

To minimize the right-hand side of the equality, one should take

$$Q_{1opt}(s) = \frac{-\theta}{1+\theta s}$$

Consequently, the optimal controller is

$$Q_{opt}(s) = \frac{\tau s - 1}{K(1+\theta s)}$$

A filter has to be introduced to satisfy the constraint for internal stability and the asymptotic tracking property:
$Q(s) = Q_{opt}(s)J(s)$. Let

$$J(s) = \frac{\beta s + 1}{\lambda s + 1}$$

where λ is the performance degree and β is a positive real number. According to (1), we have

$$\beta = \frac{\lambda\tau + \lambda\theta + 2\tau\theta}{\tau - \theta}$$

Hence,

$$\begin{aligned} C(s) &= \frac{Q(s)}{1 - G(s)Q(s)} \\ &= \frac{\beta}{K\alpha}\left(1 + \frac{1}{\beta s}\right) \end{aligned}$$

where

$$\alpha = \frac{2\theta(\lambda + \theta)}{\tau - \theta}$$

$C(s)$ is a PI controller

Relationships between close-loop response and the performance degree are shown in Figures

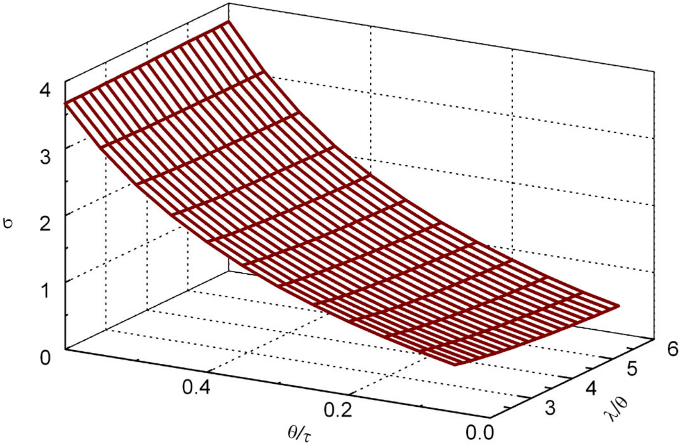

Figure: Overshoot of the H_2 control system with an unstable plant

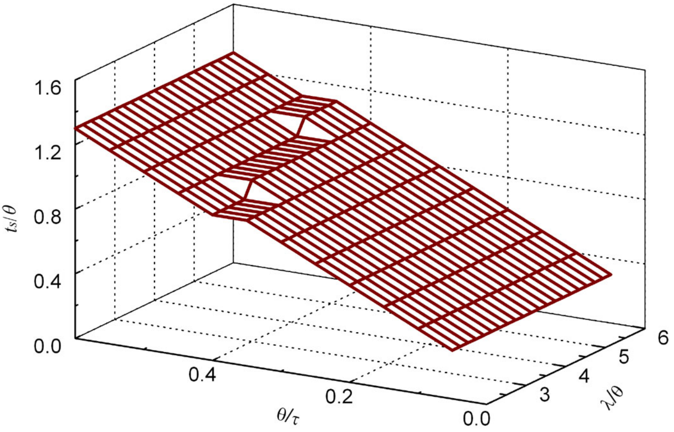

Figure: Rise time of the H_2 control system with an unstable plant

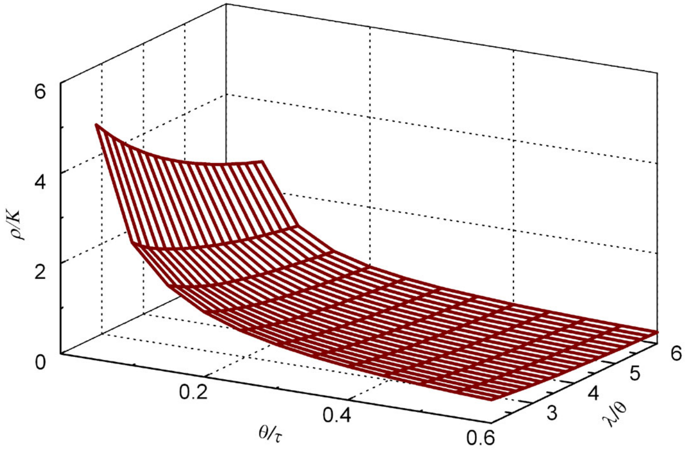

Figure: Perturbation peak of the H_2 control system with an unstable plant

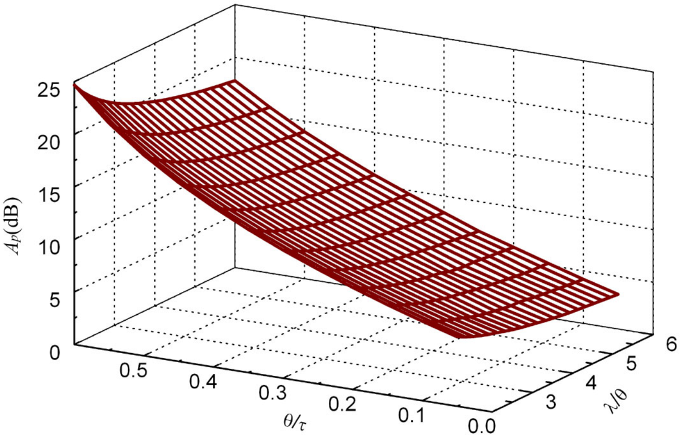

Figure: Resonance peak of the H_2 control system with an unstable plant

Design for the Second-Order Plant

Assume that the plant is of second-order:

$$G(s) = \frac{K}{(\tau_1 s - 1)(\tau_2 s + 1)} e^{-\theta s}$$

With the same design procedure, the optimal controller is obtained as follows:

$$Q_{opt}(s) = \frac{(\tau_1 s - 1)(\tau_2 s + 1)}{K(1 + \theta s)}$$

The following filter can guarantee the internal stability and asymptotic tracking:

$$J(s) = \frac{\beta s + 1}{(\lambda s + 1)^2}$$

where

$$\beta = \frac{(\lambda^2 + 2\lambda\tau_1)(\tau_1 + \theta) + 2\tau_1^2\theta}{\tau_1(\tau_1 - \theta)}$$

A little computations give

$$C(s) = \frac{1}{K}\frac{(\tau_2 s + 1)(\beta s + 1)}{s(\lambda^2\theta s/\tau_1 + \alpha)}$$

where

$$\alpha = \frac{\lambda^2(\tau_1 + \theta) + 4\lambda\tau_1\theta + 2\theta^2\tau_1}{\tau_1(\tau_1 - \theta)}$$

Compare it to the PID controller

$$C(s) = K_C\left(1 + \frac{1}{T_I s} + T_D s\right)\frac{1}{T_F s + 1}$$

the following parameters are obtained:

$$T_F = \frac{\lambda^2\theta}{\tau_1\alpha}, T_I = \tau_2 + \beta, T_D = \frac{\tau_2\beta}{\tau_2 + \beta}, K_C = \frac{\tau_2 + \beta}{K\alpha}$$

Example

A perfectly-mixed reactor is depicted in Figure, in which an exothermic, irreversible reaction takes place. Heat of reaction is removed by heat transfer to coolant in a jacket surrounding the reactor. After the reaction begins, the temperature in reaction increases with the temperature of feed. The released heat is more than the heat brought out by the coolant. Therefore, the temperature in the reactor increases and the reaction speeds up. This makes the reaction release more heat and, in return, increases the temperature in the reactor. Such a process is unstable. Why is the reactor operated at unstable working point? There are two reasons: Low temperature decreases the production rate, while high temperature is not safe and the quality of product is low. In this case, a controller is needed to guarantee the stability. The output is the temperature in the reaction. The manipulated variable is the flow rate of coolant.

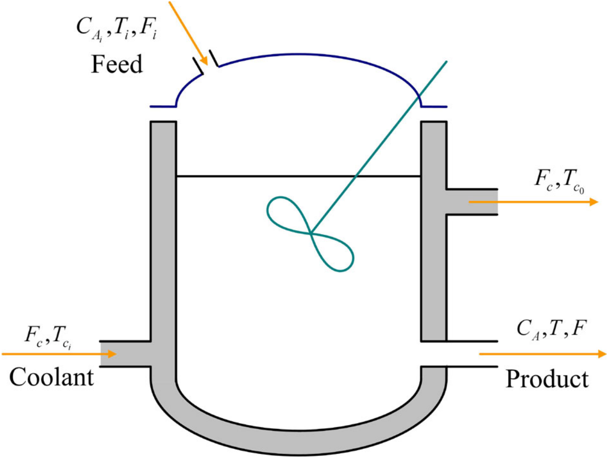

Figure: Control system of a jacket-cooled reactor

Example (ctd.1)

The dynamics of the reactor is described by

$$\boxed{G(s) = \frac{1}{s-1}e^{-0.5s}}$$

Take $\lambda = 1.5$ for the H_∞ PI controller:

$$\boxed{C(s) = \alpha\left(1 + \frac{1}{\beta s}\right)}$$

where

$$\begin{aligned} \alpha &= \frac{2\lambda^2 + 4\lambda + 1}{2(\lambda + 0.5)^2} \\ \beta &= 2\lambda^2 + 4\lambda + 1 \end{aligned}$$

Example (ctd.2)

Take $\lambda = 3$ for the H_2 PI controller:

$$\boxed{C(s) = \frac{\beta}{\alpha}\left(1 + \frac{1}{\beta s}\right)}$$

where

$$\begin{aligned} \alpha &= 2\lambda + 1 \\ \beta &= 3\lambda + 2 \end{aligned}$$

A unit step reference is added at $t = 0$ and a unit step load is added at $t = 40$. The nominal responses of the closed-loop system are shown in Figure. It is seen that the closed-loop responses have large overshoots. This is the common feature of control systems with unstable plants

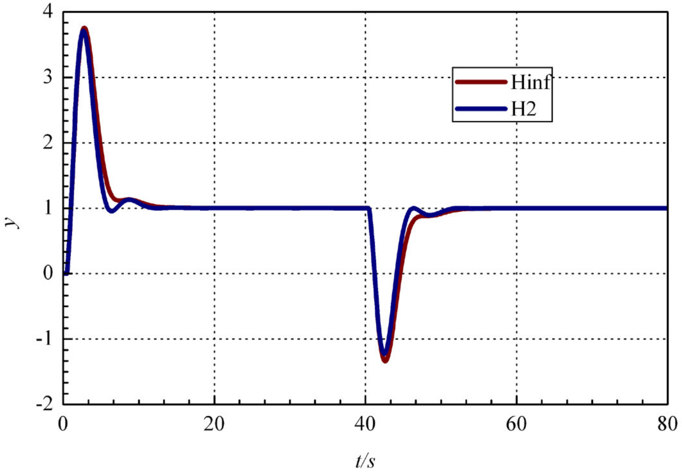

Figure: Nominal responses of the H_2 system and H_∞ system with an unstable plant

The first-order lag expansion can also be utilized to obtain the approximate model for PID controller design. Consider the first-order plant. Expand the time delay by employing the first-order lag. The approximate plant is

$$G(s) \approx \frac{K}{(\tau s - 1)(1 + \theta s)}$$

This is an MP plant. The design results of the H_∞ control and the H_2 control are the same. The optimal controller is

$$Q_{opt}(s) = \frac{(\tau s - 1)(1 + \theta s)}{K}$$

It might as well take the filter

$$J(s) = \frac{\beta s + 1}{(\lambda s + 1)^3}$$

where

$$\beta = \frac{\lambda^3}{\tau^2} + \frac{3\lambda^2}{\tau} + 3\lambda$$

Then

$$C(s) = \frac{1}{Ks}\frac{(1+\theta s)(\beta s+1)}{\lambda^3 s/\tau^2+\alpha}$$

where

$$\alpha = \frac{\lambda^3}{\tau^2} + \frac{3\lambda^2}{\tau}$$

Assume that the PID controller is in the form of

$$C(s) = K_C\left(1+\frac{1}{T_I s}+T_D s\right)\frac{1}{T_F s+1}$$

Controller parameters are

$$T_F = \frac{\lambda^3}{\tau\alpha} \qquad , T_I = \theta+\beta$$
$$T_D = \frac{\theta\beta}{\theta+\beta} \qquad , K_C = \frac{\theta+\beta}{K\alpha}$$

8.4 Performance Limitation and Robustness

Waterbed effect

Good performance implies that the maximum magnitude of $|S(j\omega)|$ in this frequency range is as small as possible. On the other hand, the maximum magnitude of $|S(j\omega)|$ over all frequencies, $\|S(j\omega)\|_\infty$, is not permitted to be too large. Unfortunately, the two aspects conflict. The situation is like a waterbed. As $|S(j\omega)|$ is pushed down in one frequency range, it pops up somewhere else

NMP plants exhibit the waterbed effect. If a plant has a zero and a pole close together in the RHP, the waterbed effect will be amplified. $|S(j\omega)|$s both in a frequency range and over all frequencies are then very large

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For example the plant

$$G(s) = \frac{K(1-\theta s)}{\tau s - 1}$$

If $\tau \to \theta$, then the zero and the pole of $G(s)$ are very close in the RHP. $G(s)$ tends to be internally unstable. It can be imagined that such a plant is very difficult to control

In what follows, the performance of the system with an unstable plant is analyzed. As introduced in Section 8.1, a general unstable plant can be described by

$$G(s) = \frac{KN_+(s)N_-(s)}{M_+(s)M_-(s)}e^{-\theta s}$$

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H_∞ control:

The quasi-H_∞ control of stable plants provide us insight into the choice of the desired closed-loop transfer function. The following desired closed-loop transfer function can be chosen:

$$T(s) = \frac{N_+(s)N_x(s)}{(\lambda s + 1)^{n_j}} e^{-\theta s}$$

where $N_x(s)$ is a polynomial with roots in the LHP, of which the order equals the number of the RHP poles of the plant, $N_x(0) = 1$, and

$$n_j = \begin{cases} \deg\{M_+\} + \{M_-\} + \{N_x\} - \{N_-\} & \{M_+\} + \{M_-\} > \{N_-\} \\ \{N_x\} + 1 & \{M_+\} + \{M_-\} = \{N_-\} \end{cases}$$

$N_x(s)$ should satisfy the constraint for the internal stability:

$$\lim_{s \to p_j} \frac{d^k}{ds^k} [1 - T(s)] = 0, k = 0, 1, ..., l_j - 1$$

or equivalently,

$$\lim_{s \to p_j} \frac{d^k}{ds^k} \left[1 - N_+(s)J(s)e^{-\theta s} \right] = 0, k = 0, 1, ..., l_j - 1$$

When $G(s)$ and $T(s)$ are known, $Q(s)$ can be derived analytically:

$$Q(s) = \frac{T(s)}{G(s)} = \frac{1}{K} \frac{M_+(s)M_-(s)N_x(s)}{N_-(s)(\lambda s + 1)^{n_j}}$$

Then the unity feedback loop controller is

$$\begin{aligned} C(s) &= \frac{T(s)}{1 - T(s)} \frac{1}{G(s)} \\ &= \frac{1}{K} \frac{M_+(s)M_-(s)N_x(s)}{N_-(s)[(\lambda s + 1)^{n_j} - N_+(s)N_x(s)e^{-\theta s}]} \end{aligned}$$

$C(s)$ contains a zero and a pole at the same point in the RHP. They must be removed by means of rational approximations

H_2 control:

For step inputs, the $Q(s)$ that makes the closed-loop system internally stable and possess the asymptotic tracking property can be described by

$$Q(s) = \frac{[1 + sQ_2(s)]M_+(s)}{K},$$

where $Q_2(s)$ is any stable transfer function that makes $Q(s)$ proper and satisfies

$$\lim_{s \to p_j} \frac{d^k}{ds^k} \left\{ 1 - \frac{[1 + sQ_2(s)]N_+(s)N_-(s)e^{-\theta s}}{M_-(s)} \right\} = 0, k = 0, 1, ..., l_j - 1.$$

The performance index is min $\|W(s)S(s)\|_2$. Then

$$\|W(s)S(s)\|_2^2 = \left\| W(s) \left\{ 1 - \frac{G(s)M_+(s)}{K} [1 + sQ_2(s)] \right\} \right\|_2^2$$

$$= \left\| \frac{M_-(s) - N_+(s)N_-(s)e^{-\theta s}}{sM_-(s)} - \frac{N_+(s)N_-(s)}{M_-(s)} e^{-\theta s} Q_2(s) \right\|_2^2$$

$$= \left\| \frac{N_+(s)}{N_+(-s)} e^{-\theta s} \left[\begin{array}{l} \frac{M_-(s)N_+(-s)e^{\theta s} - N_+(s)N_-(s)N_+(-s)}{sM_-(s)N_+(s)} - \\ \frac{N_+(-s)N_-(s)}{M_-(s)} Q_2(s) \end{array} \right] \right\|_2^2$$

$$= \left\| \begin{array}{l} \frac{M_-(s)N_+(-s)e^{\theta s} - N_+(s)N_-(s)N_+(-s)}{sM_-(s)N_+(s)} - \\ \frac{N_+(-s)N_-(s)}{M_-(s)} Q_2(s) \end{array} \right\|_2^2$$

$$= \left\| \begin{array}{l} \frac{N_+(-s)e^{\theta s} - N_+(s)}{sN_+(s)} + \\ \frac{M_-(s) - N_-(s)N_+(-s)}{sM_-(s)} - \frac{N_-(s)N_+(-s)}{M_-(s)} Q_2(s) \end{array} \right\|_2^2$$

Since $N_+(0) = M_-(0) = N_+(0)N_-(0) = 1$, s must be a factor of

$$N_+(-s)e^{\theta s} - N_+(s)$$

and

$$M_-(s) - N_-(s)N_+(-s)$$

Expand the right-hand side of the equality:

$$
\begin{aligned}
&\|W(s)S(s)\|_2^2 \\
= \ &\left\| \frac{N_+(-s)e^{\theta s} - N_+(s)}{sN_+(s)} \right\|_2^2 + \\
&\left\| \frac{M_-(s) - N_-(s)N_+(-s)}{sM_-(s)} - \frac{N_-(s)N_+(-s)}{M_-(s)} Q_2(s) \right\|_2^2
\end{aligned}
$$

Minimize the right-hand side, the optimal performance is

$$
\min \|W(s)S(s)\|_2^2 = \left\| \frac{N_+(-s)e^{\theta s} - N_+(s)}{sN_+(s)} \right\|_2^2
$$

Temporarily relax the requirement on $Q(s)$. The optimal $Q(s)$ can be derived with the help of $Q_{2opt}(s)$:

$$
Q_{opt}(s) = \frac{M_+(s)M_-(s)}{KN_-(s)N_+(-s)}
$$

To make $Q_{opt}(s)$ proper, the filter $J(s)$ is introduced: $Q(s) = Q_{opt}(s)J(s)$. Here

$$J(s) = \frac{N_x(s)}{(\lambda s + 1)^{n_j}}$$

where

$$n_j = \begin{cases} \begin{array}{l} \deg\{M_+\} + \{M_-\}+ \\ -\{N_x\}\{N_+\} - \{N_-\} \end{array} & \{M_+\} + \{M_-\} > \{N_+\} + \{N_-\} \\ \{N_x\} + 1 & \{M_+\} + \{M_-\} = \{N_+\} + \{N_-\} \end{cases}$$

$N_x(s)$ is a polynomial with roots in the LHP, of which the order equals the number of the RHP poles of the plant. $N_x(0) = 1$. $N_x(s)$ can be derived by the constraint for internal stability:

$$\lim_{s \to p_j} \frac{d^k}{ds^k} [1 - G(s)Q_{opt}(s)J(s)] = 0, k = 0, 1, ..., l_j - 1$$

or equivalently,

$$\lim_{s \to p_j} \frac{d^k}{ds^k} \left[1 - \frac{N_+(s)}{N_+(-s)} J(s) e^{-\theta s} \right] = 0, k = 0, 1, ..., l_j - 1$$

Then

$$\begin{aligned} C(s) &= \frac{Q(s)}{1 - G(s)Q(s)} \\ &= \frac{1}{K} \frac{M_+(s)M_-(s)N_x(s)}{N_-(s)[N_+(-s)(\lambda s + 1)^{n_j} - N_+(s)N_x(s)e^{-\theta s}]} \end{aligned}$$

The quasi-H_∞ controller can also be designed through the following steps:

1. If the plant does not have a time delay, turn to 3.
2. If the plant contains a time delay, take the rational part of the plant as the nominal plant.

An alternative design procedure for the quasi-H_∞ controller is as follows:

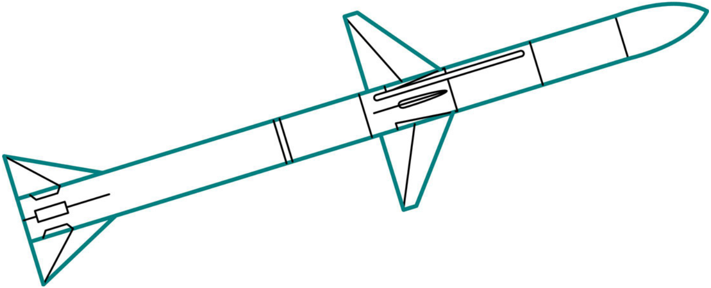
3. If the nominal plant does not have any zeros in the RHP, take its inverse as $Q_{opt}(s)$ and turn to 5.
4. If the nominal plant has zeros in the RHP, remove the factor that contains these zeros and take the inverse of the remainder as $Q_{opt}(s)$.
5. Introduce a filter to $Q_{opt}(s)$, compute the controller $C(s)$ and remove the RHP zero-pole cancellation in $C(s)$.

Design procedure for the H_2 controller is similar, except that the step 4 is modified as follows:

4. When the nominal plant has zeros in the RHP, construct an all-pass transfer function by means of the factor that contains these zeros and then remove the all-pass transfer function. Take the inverse of the remainder as $Q_{opt}(s)$.

Example

This example is used to illustrate the above design procedures for the quasi-H_∞ control and the H_2 control. A bank-to-turn missile is controlled for yaw acceleration. The input is the acceleration command and the output is the acceleration. The unit is g



Example (ctd.1)

The missile dynamics is described by

$$G(s) = \frac{-0.5(s^2 - 2500)}{(s-3)(s^2 + 50s + 1000)}$$

Normalize the plant as

$$G(s) = \frac{-5(-s/50 + 1)(s/50 + 1)}{12(-s/3 + 1)(s^2/1000 + s/20 + 1)}$$

The plant does not contain a time delay, but there is a RHP zero in it.

If a quasi-H_∞ controller is designed, the factor that contains the zero is removed and the inverse of the remainder is taken as $Q_{opt}(s)$:

Example (ctd.2)

$$Q_{opt}(s) = \frac{12(-s/3+1)(s^2/1000+s/20+1)}{-5(s/50+1)}$$

For the design of an H_2 controller, an all-pass transfer function is constructed by means of the factor containing the zero:

$$G(s) = \frac{-5(s/50+1)^2}{12(-s/3+1)(s^2/1000+s/20+1)} \frac{-s/50+1}{s/50+1}$$

The next step is to remove the all-pass transfer function and take the inverse of the remainder as $Q_{opt}(s)$:

$$Q_{opt}(s) = \frac{12(-s/3+1)(s^2/1000+s/20+1)}{-5(s/50+1)^2}$$

Robustness:

When there exists uncertainty, the analysis of the system with an unstable plant is similar to that of the system with a stable plant. The robust stability can be tested by

$$\|\Delta_m(s)T(s)\|_\infty < 1$$

Now consider the parameter uncertainty. Assume that the real plant is described by

$$\widetilde{G}(s) = \frac{\widetilde{K}e^{-\widetilde{\theta}s}}{\widetilde{\tau}s - 1}$$

where $\widetilde{K}$ is the gain, $\widetilde{\tau}$ is the time constant, $\widetilde{\theta}$ is the time delay. The three parameters are uncertain:

$$\begin{aligned} \widetilde{K} &\in [\widetilde{K}_{min}, \widetilde{K}_{max}] \\ \widetilde{\tau} &\in [\widetilde{\tau}_{min}, \widetilde{\tau}_{max}] \\ \widetilde{\theta} &\in [\widetilde{\theta}_{min}, \widetilde{\theta}_{max}] \end{aligned}$$

The nominal plant is constructed as follows:

$$G(s) = \frac{Ke^{-\theta s}}{\tau s - 1}$$

with

$$\begin{aligned} K &= \frac{\widetilde{K}_{min} + \widetilde{K}_{max}}{2} \\ \tau &= \frac{\widetilde{\tau}_{min} + \widetilde{\tau}_{max}}{2} \\ \theta &= \frac{\widetilde{\theta}_{min} + \widetilde{\theta}_{max}}{2} \end{aligned}$$

The parameter uncertainty can then be expressed as

$$\begin{aligned} |\delta K| \le \Delta K &= |\widetilde{K}_{max} - K| < |K| \\ |\delta \tau| \le \Delta \tau &= |\widetilde{\tau}_{max} - \tau| < |\tau| \\ |\delta \theta| \le \Delta \theta &= |\widetilde{\theta}_{max} - \theta| < |\theta| \end{aligned}$$

To use the test condition, one has to convert the parameter uncertainty into the unstructured uncertainty. Let the unstructured uncertain model family be

$$\widetilde{G}(s) = \frac{Ke^{-\theta s}}{\tau s - 1}\left[1 + \delta_m(s)\right]$$

where $|\delta_m(j\omega)| \leq |\Delta_m(j\omega)|$. When there are simultaneous uncertainties on the gain, the time constant, and the time delay, the following analytical expression for the unstructured uncertainty profile can be derived:

$$\Delta_m(j\omega) = \begin{cases} \left| \dfrac{|K| + \Delta K}{|K|} \dfrac{j\tau\omega - 1}{j(\tau - \Delta\tau)\omega + 1} e^{j\Delta\theta\omega} - 1 \right|, & \omega < \omega^* \\ \left| \dfrac{|K| + \Delta K}{|K|} \dfrac{j\tau\omega - 1}{j(\tau - \Delta\tau)\omega + 1} \right| + 1, & \omega \geq \omega^* \end{cases}$$

where ω^* is determined by

$$-\Delta\theta\omega^* + \arctan\frac{-\Delta\tau\omega^*}{1-\tau(-\tau+\Delta\tau)\omega^{*2}} = -\pi$$
$$\frac{\pi}{2} \le \Delta\theta\omega^* \le \pi$$

In particular, when only the gain is uncertain (that is, $\Delta\tau = \Delta\theta = 0$), the expression simplifies to

$$\Delta_m(j\omega) = \Delta K/|K|$$

When only the time constant is uncertain (that is, $\Delta K = \Delta\theta = 0$), the expression simplifies to

$$\Delta_m(j\omega) = \left|\frac{j\tau\omega - 1}{j(\tau - \Delta\tau)\omega - 1} - 1\right|$$

When only the time delay is uncertain (that is, $\Delta\tau = \Delta K = 0$), $\omega^* = \pi/\Delta\theta$. In this case,

$$\Delta_m(j\omega) = \begin{cases} \left|e^{j\Delta\theta\omega} - 1\right| & \omega < \pi/\Delta\theta \\ 2 & \omega \geq \pi/\Delta\theta \end{cases}$$

With the following tuning procedure, quantitative performance and robustness can be obtained:

Increase the performance degree monotonically until the required response is obtained

8.5 Maclaurin PID Controllers for Unstable Plants

If the RHP zero-pole cancellation in the obtained controller cannot be directly removed, a rational approximation has to be used. There are many ways to achieve this goal. In this section, the attention is paid to approximating a controller with the Maclaurin series expansion.

Consider the plant with the transfer function

$$G(s) = \frac{KN_+(s)N_-(s)}{M_+(s)M_-(s)}e^{-\theta s}$$

From the discussion in the last section, it is known that the controller designed by the quasi-H_∞ method is

$$C(s) = \frac{1}{K}\frac{M_+(s)M_-(s)N_x(s)}{N_-(s)[(\lambda s+1)^{n_j} - N_+(s)N_x(s)e^{-\theta s}]}$$

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$$C(s) = \frac{1}{K}\frac{M_+(s)M_-(s)N_x(s)}{N_-(s)[(\lambda s+1)^{n_j} - N_+(s)N_x(s)e^{-\theta s}]}$$

The controller designed by the H_2 method is

$$C(s) = \frac{1}{K} \frac{M_+(s)M_-(s)N_x(s)}{N_-(s)[N_+(-s)(\lambda s+1)^{n_j} - N_+(s)N_x(s)e^{-\theta s}]}$$

It is easy to verify that $C(s)$ has a pole at the origin. Write $C(s)$ in the form

$$C(s) = \frac{f(s)}{s}$$

The Maclaurin series expansion of $C(s)$ is

$$C(s) = \frac{1}{s}\left[f(0) + f'(0)s + \frac{f''(0)}{2!}s^2 + \ldots\right]$$

Take the first three terms to approximate the ideal controller. The three terms construct a PID controller:

$$C(s) = K_C\left(1 + \frac{1}{T_I s} + T_D s\right)$$

of which parameters are

$$K_C = f'(0), \quad T_I = \frac{f'(0)}{f(0)}, \quad T_D = \frac{f''(0)}{2f'(0)}$$

To simplify the presentation, let

$$f(s) = \frac{N(s)}{M(s)}$$

The values of $f(s)$ and its first-order and second-order derivatives at the origin can be written as

$$\begin{aligned}
f(0) &= \frac{N(0)}{M(0)} \\
f'(0) &= \frac{N'(0)M(0) - M'(0)N(0)}{M(0)^2} \\
f''(0) &= \frac{\begin{array}{c} N''(0)M(0)^2 - M''(0)N(0)M(0) - \\ 2M'(0)N'(0)M(0) + 2M'(0)^2N(0) \end{array}}{M(0)^3}
\end{aligned}$$

Consider two cases. First, assume that the plant is of first-order:

$$G(s) = \frac{K}{(\tau s - 1)} e^{-\theta s}$$

The quasi-H_∞ control and the H_2 control give the same closed-loop transfer function:

$$T(s) = \frac{(\beta s + 1)}{(\lambda s + 1)^2} e^{-\theta s}$$

By using the internal stability constraint in (1), the parameter β is obtained as follows:

$$\beta = \tau[(\lambda/\tau + 1)^2 e^{\theta/\tau} - 1]$$

Then

$$\begin{aligned} N(s) &= \frac{(\tau s - 1)(\beta s + 1)}{K}, \\ M(s) &= \frac{(\lambda s + 1)^2 - (\beta s + 1)e^{-\theta s}}{s} \end{aligned}$$

This leads to

$$\begin{aligned}
N(0) = -\frac{1}{K}, N'(0) &= \frac{\tau-\beta}{K}, N''(0) = \frac{2\tau\beta}{K}, \\
M(0) = 2\lambda+\theta-\beta, M'(0) &= \frac{2\lambda^2-\theta^2+2\beta\theta}{2}, M''(0) = \frac{\theta^3-3\beta\theta^2}{3}
\end{aligned}$$

The values of $f(s)$ and its first-order and second-order derivatives at the origin are

$$\begin{aligned}
f(0) &= -\frac{1}{K(2\lambda+\theta-\beta)} \\
f'(0) &= \frac{(\tau-\beta)(2\lambda+\theta-\beta)+(\lambda^2-\theta^2/2+\beta\theta)}{K(2\lambda+\theta-\beta)^2} \\
f''(0) &= \frac{\begin{array}{l} 2\tau\beta(2\lambda+\theta-\beta)^2+(\theta^3/3-\beta\theta^2)(2\lambda+\theta-\beta)- \\ 2(\tau-\beta)(\lambda^2-\theta^2/2+\beta\theta)(2\lambda+\theta-\beta)- \\ 2(\lambda^2-\theta^2/2+\beta\theta) \end{array}}{K(2\lambda+\theta-\beta)^3}
\end{aligned}$$

Controller parameters are as follows:

$$
\begin{aligned}
T_I &= -\tau + \beta - \frac{\lambda^2 + \beta\theta - \theta^2/2}{2\lambda + \theta - \beta} \\
K_C &= \frac{T_I}{-K(2\lambda + \theta - \beta)} \\
T_D &= \frac{-\tau\beta - (\theta^3/6 - \beta\theta^2/2)/(2\lambda + \theta - \beta)}{T_I} - \\
&\quad \frac{\lambda^2 + \beta\theta - \theta^2/2}{2\lambda + \theta - \beta}
\end{aligned}
$$

Second, the plant is of second order:

$$G(s) = \frac{K}{(\tau_1 s - 1)(\tau_2 s + 1)} e^{-\theta s}$$

Controller parameters are as follows:

$$
\begin{aligned}
T_I &= -\tau + \beta - \frac{\lambda^2 + \beta\theta - \theta^2/2}{2\lambda + \theta - \beta} \\
K_C &= \frac{T_I}{-K(2\lambda + \theta - \beta)} \\
T_D &= \frac{-\tau\beta - (\theta^3/6 - \beta\theta^2/2)/(2\lambda + \theta - \beta)}{T_I} - \\
&\quad \frac{\lambda^2 + \beta\theta - \theta^2/2}{2\lambda + \theta - \beta}
\end{aligned}
$$

Second, the plant is of second order:

$$
G(s) = \frac{K}{(\tau_1 s - 1)(\tau_2 s + 1)} e^{-\theta s}
$$

In both the quasi-H_∞ control and the H_2 control the closed-loop transfer function is

$$T(s) = \frac{\beta s + 1}{(\lambda s + 1)^3} e^{-\theta s}$$

The internal stability constraint yields

$$\beta = \tau_1[(\lambda/\tau_1 + 1)^3 e^{\theta/\tau_1} - 1]$$

Then

$$\begin{aligned} N(s) &= \frac{(\tau_1 s - 1)(\tau_2 s + 1)(\beta s + 1)}{K} \\ M(s) &= \frac{(\lambda s + 1)^3 - (\beta s + 1)e^{-\theta s}}{s} \end{aligned}$$

This leads to

$$N(0) = -\frac{1}{K}, \qquad N'(0) = \frac{\tau_1 - \tau_2 - \beta}{K}$$

$$N''(0) = \frac{2\tau_1\tau_2 + 2\tau_1\beta_1 - 2\tau_2\beta}{K}, \qquad M(0) = 3\lambda + \theta - \beta$$

$$M'(0) = \frac{6\lambda^2 - \theta^2 + 2\beta\theta}{2}, \qquad M''(0) = \frac{6\lambda^3 - 3\beta\theta^2 + \theta^3}{3}$$

The values of $f(s)$ and its first order and second order derivatives at the origin are

$$\begin{aligned} f(0) &= -\frac{1}{K(3\lambda + \theta - \beta)} \\ f'(0) &= \frac{(\tau_1 - \tau_2 - \beta)(3\lambda + \theta - \beta) + (3\lambda^2 - \theta^2/2 + \beta\theta)}{K(3\lambda + \theta - \beta)^2} \\ f''(0) &= \frac{\begin{array}{l} 2(\tau_1\tau_2 + \tau_1\beta - \tau_2\beta)(3\lambda + \theta - \beta)^2 + \\ (2\lambda^3 - \theta^2\beta + \theta^3/3)(3\lambda + \theta - \beta) - \\ 2(\tau_1 - \tau_2 - \beta)(3\lambda^2 + \theta\beta - \theta^2/2)(3\lambda + \theta - \beta) - \\ 2(3\lambda^2 + \theta\beta - \theta^2/2)^2 \end{array}}{K(3\lambda + \theta - \beta)^3} \end{aligned}$$

Then, controller parameters are

$$T_I = -\frac{(\tau_1 - \tau_2 - \beta)(3\lambda + \theta - \beta) + (3\lambda^2 + \beta\theta - \theta^2/2)}{3\lambda + \theta - \beta}$$

$$K_C = \frac{T_I}{-K(3\lambda + \theta - \beta_1)}$$

$$T_D = -\frac{\begin{array}{l} 2(\tau_1\tau_2 + \tau_1\beta - \tau_2\beta)(3\lambda + \theta - \beta)^2 + \\ (2\lambda^3 - \theta^2\beta + \theta^3/3)(3\lambda + \theta - \beta) - \\ -(\tau_1 - \tau_2 - \beta)(6\lambda^2 + 2\theta\beta - \theta^2)(3\lambda + \theta - \beta) - \\ 2(3\lambda^2 + \theta\beta - \theta^2/2)^2 \end{array}}{2T_I(3\lambda + \theta - \beta)^2}$$

8.6 PID Design for the Best Achievable Performance

Suppose the plant is described by

$$G(s) = \frac{KN_+(s)N_-(s)}{M_+(s)M_-(s)}e^{-\theta s}$$

The desired closed-loop transfer function $T(s)$ is the same as that in the last section. Then, the controller can be expressed as

$$C(s) = \frac{1}{G(s)}\frac{T(s)}{1-T(s)}$$

$C(s)$ has a pole at the origin and thus can be written as

$$C(s) = \frac{f(s)}{s}$$

The Maclaurin series expansion of $f(s)$ is

$$f(s) = f(0) + f'(0)s + \frac{f''(0)}{2!}s^2 + \frac{f^{(3)}(0)}{3!}s^3 + \ldots$$

A practical PID controller has the following expression:

$$C(s) = \frac{a_2 s^2 + a_1 s + a_0}{s(b_1 s + 1)}$$

Let the Pade approximation of $f(s)$ be

$$\frac{a_2 s^2 + a_1 s + a_0}{b_1 s + 1}$$

Then

$$\begin{aligned} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} &= \begin{bmatrix} f(0) & 0 \\ f'(0) & f(0) \\ f''(0)/2! & f'(0) \end{bmatrix} \begin{bmatrix} 1 \\ b_1 \end{bmatrix} \\ b_1 f''(0)/2! &= -f^{(3)}(0)/3! \end{aligned}$$

A little algebra yields that

$$a_0 = f(0), \qquad a_1 = b_1 f(0) + f'(0)$$

$$a_2 = b_1 f'(0) + f''(0)/2!, \qquad b_1 = -\frac{f^{(3)}(0)}{3f''(0)}$$

If the practical PID controller is in the form of

$$C = K_C\left(1 + \frac{1}{T_I s} + T_D s\right)\frac{1}{T_F s + 1}$$

Controller parameters are

$$K_C = a_1, \quad T_I = \frac{a_1}{a_0}, \quad T_D = \frac{a_2}{a_1}, \quad T_F = b_1$$

All these parameters should be positive.

Consider the first-order unstable plant:

$$G(s) = \frac{Ke^{-\theta s}}{\tau s - 1}$$

The values of $f(s)$ and its first order-third order derivatives are

$$f(0) = -\frac{1}{K(3\lambda + \theta - \beta)}$$

If the practical PID controller is in the form of

$$C = K_C \left(1 + \frac{1}{T_I s} + T_D s\right) \frac{1}{T_F s + 1}$$

Controller parameters are

$$K_C = a_1, \quad T_I = \frac{a_1}{a_0}, \quad T_D = \frac{a_2}{a_1}, \quad T_F = b_1$$

All these parameters should be positive.

Consider the first-order unstable plant:

$$G(s) = \frac{Ke^{-\theta s}}{\tau s - 1}$$

The values of $f(s)$ and its first order-third order derivatives are

$$f(0) = -\frac{1}{K(3\lambda + \theta - \beta)}$$

$$f'(0) = \frac{(\tau_1 - \tau_2 - \beta)(3\lambda + \theta - \beta) + (3\lambda^2 - \theta^2/2 + \beta\theta)}{K(3\lambda + \theta - \beta)^2}$$

$$f''(0) = \frac{\begin{array}{l} 2(\tau_1\tau_2 + \tau_1\beta - \tau_2\beta)(3\lambda + \theta - \beta)^2 + \\ (2\lambda^3 - \theta^2\beta + \theta^3/3)(3\lambda + \theta - \beta) - \\ 2(\tau_1 - \tau_2 - \beta)(3\lambda^2 + \theta\beta - \theta^2/2)(3\lambda + \theta - \beta) - \\ 2(3\lambda^2 + \theta\beta - \theta^2/2)^2 \end{array}}{K(3\lambda + \theta - \beta)^3}$$

$$f^{(3)}(0) = \frac{\begin{array}{l} -6\beta\tau(\lambda^2 - \theta^2/2 + \beta\theta)(2\lambda + \theta - \beta)^2 + \\ 6(\tau - \beta)(\lambda^2 - \theta^2/2 + \beta\theta)^2(2\lambda + \theta - \beta) - \\ 3(\tau - \beta)(\theta^3/3 - \beta\theta^2)(2\lambda + \theta - \beta)^2 - \\ (2\theta^3 - 6\beta\theta^2)(\lambda^2 - \theta^2/2 + \beta\theta)(2\lambda + \theta - \beta) + \\ (\beta\theta^3 - \theta^4/4)(2\lambda + \theta - \beta)^2 + \\ 6(\lambda^2 - \theta^2/2 + \beta\theta)^3 \end{array}}{K(2\lambda + \theta - \beta)^4}$$

Clearly, parameters of the PID controller are

$$
\begin{aligned}
T_F &= -\frac{f^{(3)}(0)}{3f''(0)} \\
K_C &= T_F f(0) + f'(0) \\
T_I &= \frac{K_C}{f(0)} \\
T_D &= \frac{T_F f'(0) + f''(0)/2!}{K_C}
\end{aligned}
$$

The quantitative performance and roustness can also be obtained by monotonically increasing λ

8.7 All Stabilizing PID Controllers for Unstable Plants

Question: The closed-loop system is open-loop during $t < \theta$. If the plant is stable, there is no problem. Nevertheless, if the plant is unstable, the system output will continuously increase until the physical limitation is reached. This may cause such a problem. When $t < \theta$, the system output becomes very large. The PID controller does not act until $t = \theta$. Then whether can the feedback control pull the system output back to a new equilibrium? Furthermore, are there any stabilizing PID controllers?This section will discuss the question.

Similar to the discussion for stable plants, the attention is restricted to the first-order plant with time delay:

$$G(s) = \frac{Ke^{-\theta s}}{\tau s - 1}$$

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Similar to the discussion for stable plants, the attention is restricted to the first-order plant with time delay:

$$G(s) = \frac{Ke^{-\theta s}}{\tau s - 1}$$

and the standard PID controller:

$$C = K_C + \frac{K_I}{s} + K_D s$$

Theorem

If $\theta \geq 2\tau$, there does not exist a stabilizing PID controller for the first-order unstable plant with time delay.

Proof.

The following quasi-polynomial can be used to analyze the stability of the closed loop system:

$$\delta^*(s) = -K(K_I + K_C s + K_D s^2) + (1 - \tau s)se^{\theta s}$$

Proof ctd.1.

The imaginary part of $\delta^*(j\omega)$ is

$$\delta_i(\omega) = \omega[-KK_C + \cos(\theta\omega) + \tau\omega\sin(\theta\omega)]$$

Define the following function:

$$f(z, K_C) = \frac{-KK_C + \cos(z)}{\sin(z)}$$

where $z = \theta\omega$. To prove the theorem, it is sufficient to prove that the roots of $\delta_i(\omega)$ are not all real for $\theta \geq 2\tau$, or equivalently, $f(z, K_C)$ and the line $-\tau z/\theta$ do not intersect in $z \in (0, \pi)$. It was discussed in the proof of Theorem 4.6.1 that such a case implied instability.

Proof ctd.2.

Consider $K_{C1} < K_{C2}$. For any $z \in (0, \pi)$,

$$-KK_{C1} + \cos(z) > -KK_{C2} + \cos(z)$$

Since $\sin(z) > 0$,

$$f(z, K_{C1}) > f(z, K_{C2})$$

In other words, for any fixed $z \in (0, \pi)$, $f(z, K_C)$ is monotonically decreasing with respect to the increase of K_C. Hence, for $K_C > 1/K$ and any $z \in (0, \pi)$

$$f(z, K_C) < f(z, \frac{1}{K})$$

Proof ctd.3.

This implies that if the line $-\tau z/\theta$ does not intersect the curve $f(z, 1/K)$ in $z \in (0, \pi)$, it will not intersect any other curve $f(z, K_C)$ in $z \in (0, \pi)$.
It is observed that for any $z \in (0, \pi)$

$$f(z, \frac{1}{K}) = \frac{-1 + \cos(z)}{\sin(z)} = -\tan\left(\frac{z}{2}\right)$$

Accordingly, define an extension of $f(z, 1/K)$ over $[0, \pi)$ by

$$f_1(z, \frac{1}{K}) = -\tan\left(\frac{z}{2}\right)$$

Clearly, the curve $f_1(z, 1/K)$ intersects the line $-\tau z/\theta$ at $z = 0$ (Figure). Also, it is observed that the slope of the tangent to $f_1(z, 1/K)$ at $z = 0$ is given by

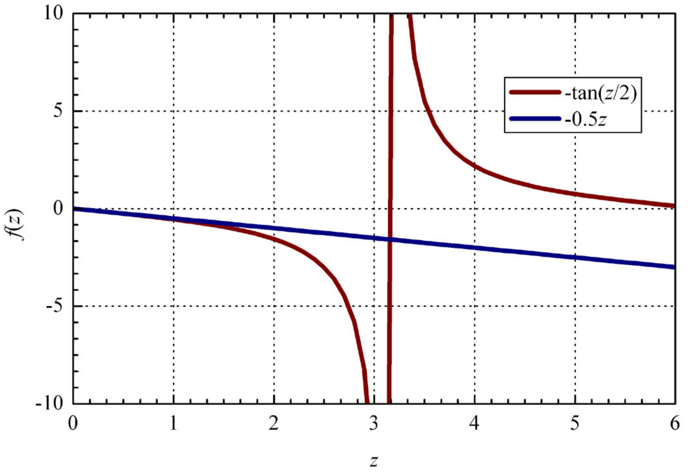

Figure: Plots of the curve $f_1(z, 1/K)$ and the line $-\tau z/\theta$ (From Silva et al., 2002. Reprinted by permission of the IEEE)

Proof ctd.4.

$$\frac{df_1}{dz} = -\frac{1}{2}\sec^2\frac{z}{2}\bigg|_{z=0} = -\frac{1}{2}$$

If this slope is less than or equal to $-\tau/\theta$, then no further intersections will take place over $(0, \pi)$. Since $f(z, 1/K) = f_1(z, 1/K)$ in $(0, \pi)$, the curve $f(z, 1/K)$ will not intersect the line $-\tau z/\theta$ for $\theta \geq 2\tau$. This completes the proof

Now, assume that the condition in Theorem is satisfied. When the closed-loop system is stable, what range should the PID controller parameters be in? The following theorem gives the answer

Proof ctd.4.

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Now, assume that the condition in Theorem is satisfied. When the closed-loop system is stable, what range should the PID controller parameters be in? The following theorem gives the answer

Theorem

If $\theta < 2\tau$, then the unstable plant can be stabilized by a PID controller if and only if

$$\frac{1}{K} < K_C < K_T$$

where

$$K_T = \frac{1}{K}\left[\frac{-\tau}{\theta}\alpha_1 \sin(\alpha_1) + \cos(\alpha_1)\right]$$

and α_1 is the solution of the equation

$$\tan(\alpha) = \frac{\tau}{\tau + \theta}\alpha$$

in the interval $(0, \pi)$. In particular, when $\theta = \tau$, $\alpha_1 = \pi/2$.

Theorem (ctd.1)

Furthermore, for each $K_C \in (1/K, K_T)$, the stabilizing region of the integral constant and the derivative constant is the quadrilateral in Figure. Here

$$\begin{aligned} m(z) &= \frac{\theta^2}{z^2} \\ b(z) &= \frac{\theta}{Kz}\left[\sin(z) - \frac{\tau}{\theta} z \cos(z)\right] \\ w(z) &= -\frac{z}{K\theta}\left\{\sin(z) - \frac{\tau}{\theta} z[\cos(z) + 1]\right\} \end{aligned}$$

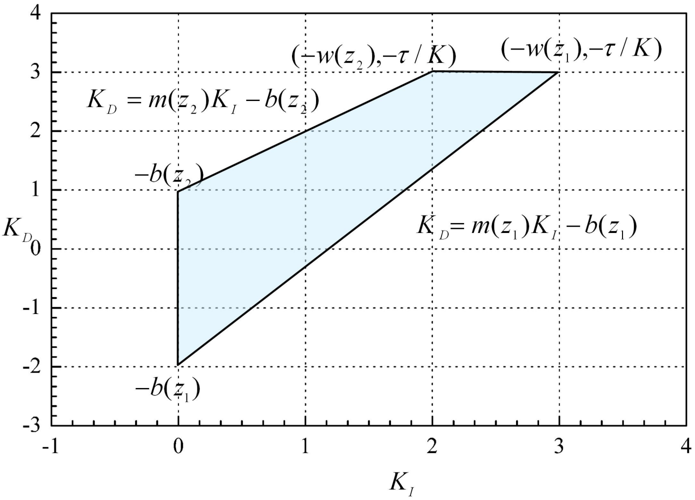
and $z_j (j = 1, 2, ...)$ are the positive real roots of

$$-KK_C + \cos(z) + \frac{\tau}{\theta} z \sin(z) = 0.$$

These roots are arranged in increasing order of magnitude.

Proof.

The proof is similar to Theorem 4.6.1 and thus omitted here. □



End of Chapter 8