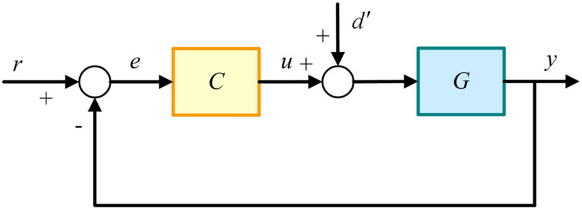
Chapter 7 Control of Integrating Plants

7.1 The Feature of Integrating Systems

Assumption: Integrating plants in this book do not have any open RHP poles. Those with poles in the open RHP are included in unstable plants. **This assumption is made solely for simplicity of presentation**

Consider the feedback control loop in Figure, where $G(s)$ is an integrating plant and $C(s)$ is the controller



Internal Stability

The closed-loop system is internally stable if and only if all elements in the transfer matrix $\mathbf{H}(s)$are stable:

$$\begin{bmatrix} y(s) \\ u(s) \end{bmatrix} = \mathbf{H}(s) \begin{bmatrix} r(s) \\ d'(s) \end{bmatrix}$$

where

$$\mathbf{H}(s) = \begin{bmatrix} \dfrac{G(s)C(s)}{1+G(s)C(s)} & \dfrac{G(s)}{1+G(s)C(s)} \\ \dfrac{C(s)}{1+G(s)C(s)} & \dfrac{-G(s)C(s)}{1+G(s)C(s)} \end{bmatrix}$$

Since the Youla parameterization for stable plants cannot be used for integrating plants, the following transfer function is defined:

$$Q(s) = \frac{C(s)}{1+G(s)C(s)}$$

The transfer function $Q(s)$ is in fact the IMC controller. Then the transfer matrix $\mathbf{H}(s)$ becomes

$$\mathbf{H}(s) = \begin{bmatrix} G(s)Q(s) & [1 - G(s)Q(s)]G(s) \\ Q(s) & -G(s)Q(s) \end{bmatrix}$$

Since $G(s)$ is not stable, the stability of $Q(s)$ cannot guarantee the stability of the closed-loop system.

Theorem

Assume that $G(s)$ is an integrating plant. The unity feedback loop shown in Figure is internally stable if and only if

1. *$Q(s)$ is stable.*
2. *$[1 - G(s)Q(s)]G(s)$ is stable.*

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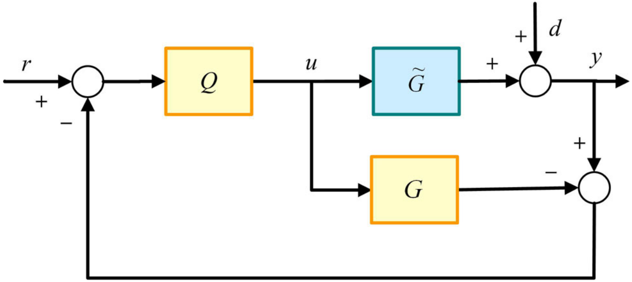
1. *$Q(s)$ is stable.*
2. *$[1 - G(s)Q(s)]G(s)$ is stable.*

Proof.

Necessity is obvious. Consider sufficiency. Assume that the two conditions hold. It remains to show that $G(s)Q(s)$ is stable. If $G(s)Q(s)$ is unstable, $1 - G(s)Q(s)$ is unstable, which implies that $[1 - G(s)Q(s)]G(s)$ must be unstable. This contradicts the assumption. □

The conclusion may not be applicable to **other structures**. Consider the IMC structure shown in Figure



When the model is exact, the system is open-loop for $G(s)$ and $Q(s)$. Since $G(s)$ is unstable and $G(s)Q(s)$ is stable, there must exist closed RHP zero-pole cancellation between $G(s)$ and $Q(s)$. In this case, the closed-loop system is not internally stable

Consequently, the IMC structure cannot be used for the control of integrating plants

Steady-state Performance

Consider the first-order integrating plant:

$$G(s) = \frac{K}{s}e^{-\theta s}$$

where K is the gain, θ is the time delay. Assume that the disturbance at the plant input is $d'(s) = 1/s$. The effect of $d'(s)$ on the system output can be equivalent to that of a disturbance $d(s)$ at the plant output:

$$d(s) = d'(s)G(s) = \frac{K}{s^2}e^{-\theta s}$$

It is seen that the system is in fact of **Type 2**. Only when the controller is designed for ramps, can the steady-state error caused by $d'(s)$ vanish asymptotically

In general, if the plant has m poles at the origin, the system should be of Type $m+1$ for asymptotic tracking; or equivalently, the controller has to satisfy

$$\lim_{s\to 0}\frac{1-G(s)Q(s)}{s^k}=0, k=0,1,...,m$$

or

$$\lim_{s\to 0}\frac{d^k}{ds^k}[1-G(s)Q(s)]=0, k=0,1,...,m$$

This conclusion is very important in the design of systems with integrating plants

Derivatives of a function are frequently calculated in the design of systems with integrating plants. To avoid complicated computation, two algebra results are given here

Theorem

The kth $(k = 0, 1, ..., q)$ coefficient of a q-order polynomial $N(s)$ is $d^k N(0)/ds^k/k!$.

Proof.

Follows directly from its Taylor series expansion. □

Theorem

Given the transfer function $N(s)/M(s)$. $N(s)$ and $M(s)$ are polynomials, and $q = \deg\{N(s)\} \leq p = \deg\{M(s)\}$. Let $m \leq q$ be any nonnegative integer. Then

$$\lim_{s \to 0} \frac{d^k}{ds^k} \left[1 - \frac{N(s)}{M(s)} \right] = 0, k = 0, 1, ..., m$$

holds if and only if the coefficients of the first $k (k = 0, 1, ..., m)$ terms of $N(s)$ are the same as those of $M(s)$, respectively.

Proof.

Sufficiency is obvious. To prove necessity, assume that

$$N(s) = \beta_q s^q + ... + \beta_k s^k + ... + \beta_1 s + \beta_0,$$
$$M(s) = \alpha_p s^p + ... + \alpha_k s^k + ... + \alpha_1 s + \alpha_0,$$

where $\beta_i (i = 0, 1, ..., q)$ and $\alpha_i (i = 0, 1, ..., p)$ are positive real numbers. Let

$$F(s) = 1 - \frac{N(s)}{M(s)}.$$

Then

$$M(s)F(s) = M(s) - N(s).$$

Proof ctd.1.

The inductive method is used: First, the case $k = 0$ and $k = 1$ are shown to be true; Then the case for the k-order is shown to be true if the case for the $(k-1)$-order is true.

When $k = 0$,

$$\lim_{s \to 0} F(s) = \frac{\alpha_0 - \beta_0}{\alpha_0}$$

Let the right-hand side be 0. We have $\alpha_0 = \beta_0$

When $k = 1$,

$$\begin{aligned} \frac{d}{ds}[M(s)F(s)] &= F(s)\frac{d}{ds}M(s) + M(s)\frac{d}{ds}F(s) \\ \frac{d}{ds}[M(s) - N(s)] &= \frac{d}{ds}M(s) - \frac{d}{ds}N(s) \end{aligned}$$

Proof ctd.2.

Since

$$\lim_{s\to 0} F(s) = 0,$$

the derivative of $F(s)$ is

$$\begin{aligned}\lim_{s\to 0} \frac{d}{ds}F(s) &= \frac{\frac{d}{ds}M(s) - \frac{d}{ds}N(s)}{M(s)} \\ &= \frac{\alpha_1 - \beta_1}{\alpha_0}.\end{aligned}$$

Let $\lim_{s\to 0} \frac{d}{ds}F(s) = 0$. This yields

$$\alpha_1 = \beta_1.$$

Proof ctd.3.

Now assume that the conclusion holds for $k-1$. To prove the theorem, it suffices to prove that the conclusion holds for the kth time differentiating. Consider the following fact:

$$\begin{aligned}
\frac{d^k}{ds^k}[M(s)F(s)] &= \frac{d^k}{ds^k}M(s)F(s) + C_k^1\frac{d^{k-1}}{ds^{k-1}}M(s)\frac{d}{ds}F(s) + \ldots \\
&\quad + C_k^{k-1}\frac{d}{ds}M(s)\frac{d^{k-1}}{ds^{k-1}}F(s) + M(s)\frac{d^k}{ds^k}F(s) \\
\frac{d^k}{ds^k}[M(s)-N(s)] &= \frac{d^k}{ds^k}M(s) - \frac{d^k}{ds^k}N(s),
\end{aligned}$$

where

$$C_k^i = \frac{k!}{i!(k-i)!}.$$

Proof ctd.4.

With the assumption, we have

$$\lim_{s\to 0}\frac{d^k}{ds^k}F(s) = \frac{\frac{d^k}{ds^k}M(s) - \frac{d^k}{ds^k}N(s)}{M(s)} = \frac{\alpha_k - \beta_k}{\alpha_0}.$$

The left-hand side should be 0. Therefore,

$$\alpha_k = \beta_k.$$

This completes the proof. □

Corollary

Given the transfer function $N(s)/M(s)$.

$$\lim_{s\to 0}\left[1-\frac{N(s)}{M(s)}\right]=0 \text{ and } \lim_{s\to 0}\frac{d}{ds}\left[1-\frac{N(s)}{M(s)}\right]=0$$

hold if and only if the coefficients of the first 2 terms of $N(s)$ are the same as those of $M(s)$, respectively.

If a quasi-polynomial that contains a time delay is encountered when using these results, the time delay should be substituted by its Taylor series expansion.

Example

There are two polynomials: $N(s) = (1 - \theta s/2)(\beta_1 s + 1)$ and $M(s) = (1 + \theta s/2)(\lambda s + 1)^2$. Compute the constant β_1 that makes the following hold:

$$\lim_{s \to 0} \left[1 - \frac{N(s)}{M(s)}\right] = 0 \text{ and } \lim_{s \to 0} \frac{d}{ds} \left[1 - \frac{N(s)}{M(s)}\right] = 0$$

According to Corollary, the zeroth-order and the first-order coefficients of $N(s)$ and $M(s)$ should equal, respectively. Both the zeroth-order coefficients of $N(s)$ and $M(s)$ are 1. The first-order coefficient of $N(s)$ is $\beta_1 - \theta/2$ and the first-order coefficient of $M(s)$ is $2\lambda + \theta/2$. This yields

$$\beta_1 = 2\lambda + \theta$$

Example

It is known that $N(s) = (\beta_1 s + 1)e^{-\theta s}$ and $M(s) = (\lambda s + 1)^{n_j}$. Compute the constant β_1 that makes

$$\lim_{s \to 0} \left[1 - \frac{N(s)}{M(s)} \right] = 0 \text{ and } \lim_{s \to 0} \frac{d}{ds} \left[1 - \frac{N(s)}{M(s)} \right] = 0$$

hold.

Again, the zeroth-order and the first-order coefficients of $N(s)$ and $M(s)$ should equal, respectively. Both the zeroth-order coefficients of $N(s)$ and $M(s)$ are 1. The first-order coefficient of $N(s)$ is $\beta_1 - \theta$ and the first-order coefficient of $M(s)$ is $n_j\lambda$. Let them equal. One readily obtains

$$\beta_1 = n_j\lambda + \theta$$

7.2 H_∞ PID Controllers for Integrating Plants

One Way to Design H_∞ PID Controllers

Assume that the coprime factorization of $G(s)$ is $G(s) = V(s)/U(s)$, where $U(s)$ and $V(s)$ are stable proper real rational. According to the discussion in Section 3.3, all stabilizing controllers for integrating plants can be expressed as

$$C(s) = \frac{X(s) + U(s)Q(s)}{Y(s) - V(s)Q(s)}$$

where $Q(s)$ is stable, and $X(s)$ and $Y(s)$ are stable proper real rational functions that satisfy the equation

$$V(s)X(s) + U(s)Y(s) = 1$$

The design procedure is as follows:

1. Expand the time delay by the Pade approximation
2. Take the performance index as min $\|W(s)S(s)\|_\infty$
3. Design the controller by following steps:
   1. Calculate the coprime factorization of the plant: $G(s) = V(s)/U(s)$. Then $S(s) = U(s)[Y(s) - V(s)Q(s)]$
   2. Derive $Q_{opt}(s)$ by minimizing $\|W(s)S(s)\|_\infty$
   3. Introduce a filter to roll $Q_{opt}(s)$ off at high frequencies.
   4. Compute the controller $C(s)$ by $Q(s)$

Such a design procedure will not be adopted in this section, since it is tedious to obtain a coprime factorization. There are only **numerical algorithms** available. A simple design procedure is developed here

Simplified Design

1. Following the Youla parameterization of stable plants a transfer function $Q(s)$ is defined:

$$Q(s) = \frac{C(s)}{1 + G(s)C(s)}$$

   This was already done in the last section. To guarantee the internal stability, $Q(s)$ has to satisfy that

   1. $Q(s)$ is stable
   2. $G(s)[1 - G(s)Q(s)]$ is stable

2. Design the optimal controller $Q_{opt}(s)$ for step inputs
3. Introduce an appropriate filter for the internal stability and the asymptotic tracking property, and compute $C(s)$

Design for the First-Order Plant

Consider the first-order integrating process:

$$G(s) = \frac{K}{s} e^{-\theta s}$$

An approximate plant is obtained by employing the 1/1 Pade approximant:

$$G(s) \approx \frac{K(1 - \theta s/2)}{s(1 + \theta s/2)}$$

The internal stability requires that

$$\lim_{s \to 0} [1 - G(s)Q(s)] = 0$$

To satisfy the condition, s must be a factor of $Q(s)$, and the constant term of the remainder must be $1/K$

that is,

$$Q(s) = \frac{s[1 + sQ_1(s)]}{K}$$

where $Q_1(s)$ is a stable transfer function
Assume that the input is a unit step. Take $W(s) = 1/s$. The approximate plant has a RHP zero at $2/\theta$. Then

$$\|W(s)S(s)\|_\infty = \|W(s)[1 - G(s)Q(s)]\|_\infty \geq |W(2/\theta)|$$

Minimizing the left-hand side of the equality yields

$$\min \left\| W(s) \left\{ 1 - G(s) \frac{s[1 + sQ_1(s)]}{K} \right\} \right\|_\infty = \theta/2$$

Then the optimal controller is

$$Q_{1opt}(s) = \theta/2$$

This yields

$$Q_{opt}(s) = \frac{s}{K}\left(1 + \frac{\theta}{2}s\right)$$

Similar to the design for stable plants, a filter is introduced to $Q_{opt}(s)$: $Q(s) = Q_{opt}(s)J(s)$. For asymptotic tracking, the system with the first-order integrating plant has to be of Type 2, which imposes the following constraint on $Q(s)$:

$$\lim_{s\to 0}[1 - G(s)Q(s)] = 0$$

$$\lim_{s\to 0}\frac{d}{ds}[1 - G(s)Q(s)] = 0$$

To achieve this, the filter has a more complex form than that in the system with a stable plant. It must have a zero:

$$J(s) = \frac{\beta s + 1}{(\lambda s + 1)^{n_j}}$$

where λ is the performance degree, n_j should be large enough to make $Q(s)$ bi-proper, and β is a positive real number and chosen to satisfy the asymptotic tracking constraint

It is easy to verify that a first-order or second-order filter cannot satisfy the asymptotic tracking requirement. It might as well choose the third-order filter. The filter satisfying the requirement can be obtained as follows:

$$J(s) = \frac{(3\lambda + \theta/2)s + 1}{(\lambda s + 1)^3}$$

The suboptimal controller is

$$Q(s) = \frac{s(1 + \theta s/2)\left[(3\lambda + \theta/2)s + 1\right]}{K(\lambda s + 1)^3}$$

A little algebra gives

$$\begin{aligned} C(s) &= \frac{Q(s)}{1-G(s)Q(s)} \\ &= \frac{1}{K}\frac{s(1+\theta s/2)\left[(3\lambda+\theta/2)s+1\right]}{(\lambda s+1)^3-(1-\theta s/2)\left[(3\lambda+\theta/2)s+1\right]} \\ &= \frac{1}{K}\frac{\left(3\lambda\theta/2+\theta^2/4\right)s^2+(3\lambda+\theta)s+1}{\lambda^3 s^2+\left(3\lambda^2+3\lambda\theta/2+\theta^2/4\right)s} \end{aligned}$$

This is a PID . Compare it with the following PID:

$$C(s) = K_C\left(1+\frac{1}{T_I s}+T_D s\right)\frac{1}{T_F s+1}$$

Controller parameters are

$$T_F = \frac{\lambda^3}{3\lambda^2+3\lambda\theta/2+\theta^2/4}, \qquad T_I = 3\lambda+\theta$$

$$T_D = \frac{3\lambda\theta/2+\theta^2/4}{T_I}, \qquad K_C = \frac{1}{K}\frac{T_I}{3\lambda^2+3\lambda\theta/2+\theta^2/4}$$

The controller can also be tuned for quantitative responses

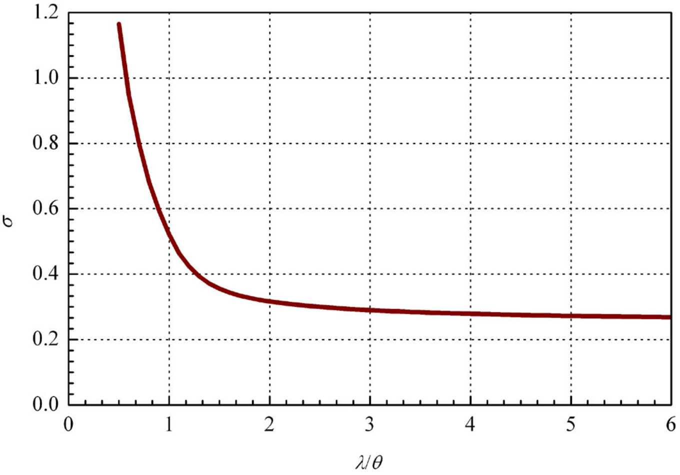


Figure: Overshoot of the H_∞ control system with an integrating plant

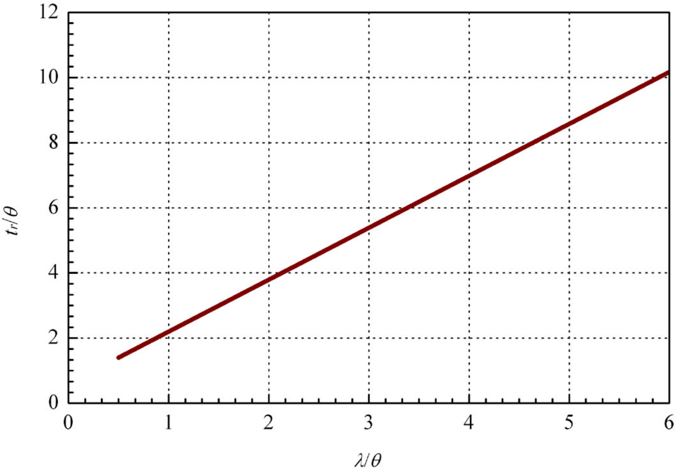


Figure: Rise time of the H_∞ control system with an integrating plant

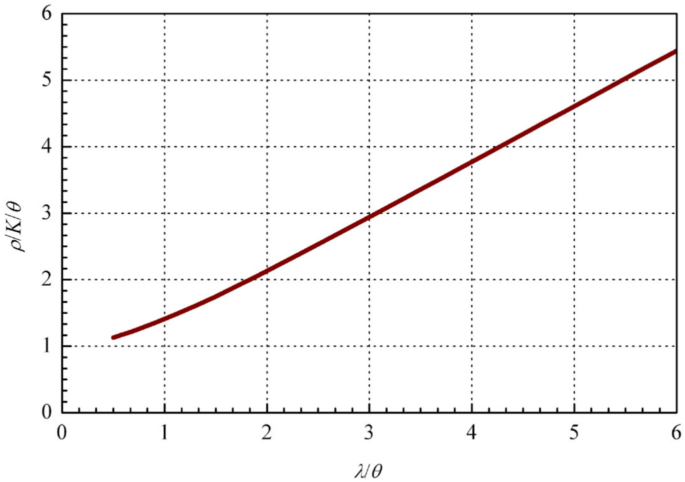


Figure: Perturbation peak of the H$_\infty$ control system with an integrating plant

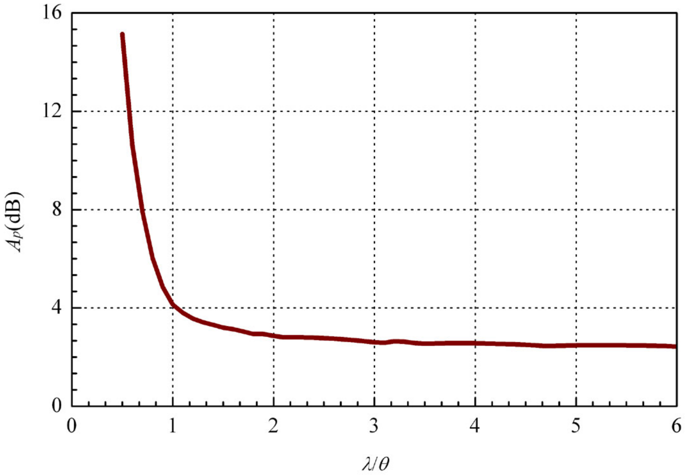


Figure: Resonance peak of the H_∞ control system with an integrating plant

Design for the Second-Order Plant

Assume the second-order integrating plant is expressed as

$$G(s) = \frac{K}{s(\tau s + 1)} e^{-\theta s}$$

where τ is the time constant. With the first-order Taylor series expansion, the plant can be rewritten as follows:

$$G(s) = \frac{K(1 - \theta s)}{s(\tau s + 1)}$$

Apply the H$_\infty$ design procedure. The solution is

$$Q_{opt}(s) = \frac{s(\tau s + 1)}{K}$$

To get a proper $Q(s)$, the following filter is introduced:

$$J(s) = \frac{(3\lambda + \theta)s + 1}{(\lambda s + 1)^3}$$

Then the unity feedback loop controller is

$$C(s) = \frac{1}{K} \frac{(\tau s + 1)[(3\lambda + \theta)s + 1]}{s(\lambda^3 s + 3\lambda^2 + 3\lambda\theta + \theta^2)}$$

If the PID controller is in the form of

$$C(s) = K_C \left(1 + \frac{1}{T_I s} + T_D s\right) \frac{1}{T_F s + 1}$$

The following controller parameters can readily be obtained:

$$T_F = \frac{\lambda^3}{3\lambda^2 + 3\lambda\theta + \theta^2}, \qquad T_I = 3\lambda + \theta + \tau$$

$$T_D = \frac{(3\lambda + \theta)\tau}{T_I}, \qquad K_C = \frac{1}{K} \frac{T_I}{3\lambda^2 + 3\lambda\theta + \theta^2}$$

7.3 H_2 PID Controllers for Integrating Plants

Design for the First-Order Plants

Consider the approximate first-order integrating plant obtained by utilizing the 1/1 Pade approximant:

$$G(s) \approx \frac{K(1-\theta s/2)}{s(1+\theta s/2)}$$

The performance index is chosen as $\min \|W(s)S(s)\|_2$. The internal stability requires that

$$\lim_{s\to 0}[1-G(s)Q(s)] = 0$$

The $Q(s)$ that satisfies the requirement can be expressed as

$$Q(s) = \frac{s[1+sQ_1(s)]}{K}$$

where $Q_1(s)$ is stable

This leads to

$$
\begin{aligned}
& \|W(s)S(s)\|_2^2 \\
= & \left\| \frac{1}{s}\left\{1 - \frac{1-\theta s/2}{1+\theta s/2}[1+sQ_1(s)]\right\}\right\|_2^2 \\
= & \left\| \frac{\theta}{1+\theta s/2} - \frac{1-\theta s/2}{1+\theta s/2}Q_1(s)\right\|_2^2 \\
= & \left\| \frac{\theta}{1-\theta s/2} - Q_1(s)\right\|_2^2 \\
= & \left\| \frac{\theta}{1-\theta s/2}\right\|_2^2 + \|Q_1(s)\|_2^2
\end{aligned}
$$

Evidently, $Q_{1opt}(s) = 0$ gives the optimal solution, which implies

$$Q_{opt}(s) = s/K$$

To satisfy the constraints for asymptotic tracking

$$\lim_{s\to 0}[1 - G(s)Q(s)] = 0 \text{ and } \lim_{s\to 0}\frac{d}{ds}[1 - G(s)Q(s)] = 0$$

introduce the following filter:

$$J(s) = \frac{(2\lambda + \theta)s + 1}{(\lambda s + 1)^2}$$

The suboptimal solution is $Q(s) = Q_{opt}(s)J(s)$. The unity feedback loop controller is

$$\begin{aligned} C(s) &= \frac{Q(s)}{1 - G(s)Q(s)} \\ &= \frac{1}{K}\frac{s[(2\lambda + \theta)s + 1](1 + \theta s/2)}{(\lambda s + 1)^2(1 + \theta s/2) - (1 - \theta s/2)[(2\lambda + \theta)s + 1]} \\ &= \frac{1}{K}\frac{(1 + \theta s/2)[(2\lambda + \theta)s + 1]}{\theta\lambda^2 s^2/2 + (\lambda^2 + 2\lambda\theta + \theta^2/2)s} \end{aligned}$$

Compare the obtained controller with the PID controller

$$C(s) = K_C\left(1 + \frac{1}{T_I s} + T_D s\right)\frac{1}{T_F s + 1}$$

Controller parameters are

$$T_F = \frac{\lambda^2\theta}{2\lambda^2 + 4\lambda\theta + \theta^2}, \qquad T_I = 2\lambda + \frac{3\theta}{2}$$

$$T_D = \frac{(2\lambda + \theta)\theta}{2T_I}, \qquad K_C = \frac{1}{K}\frac{T_I}{\lambda^2 + 2\lambda\theta + \theta^2/2}$$

Relationships between the performance degree and the closed-loop responses are shown in Figures

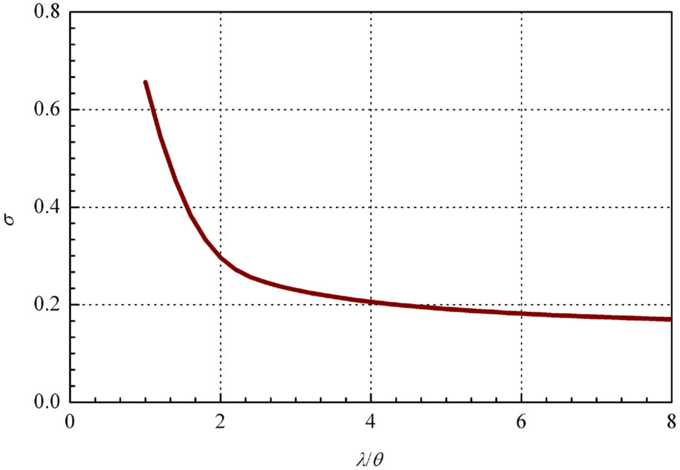


Figure: Overshoot of the H_2 control system with an integrating plant

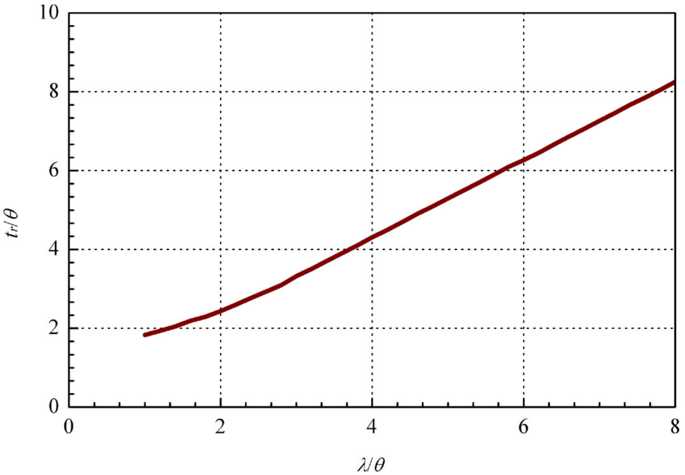


Figure: Rise time of the H_2 control system with an integrating plant

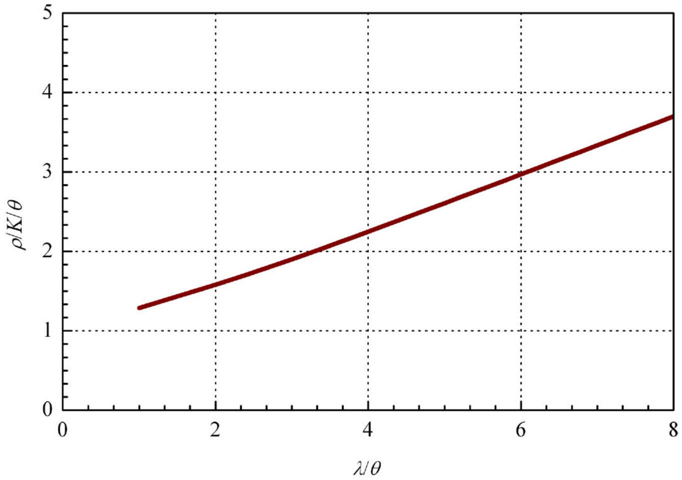


Figure: Perturbation peak of the H_2 control system with an integrating plant

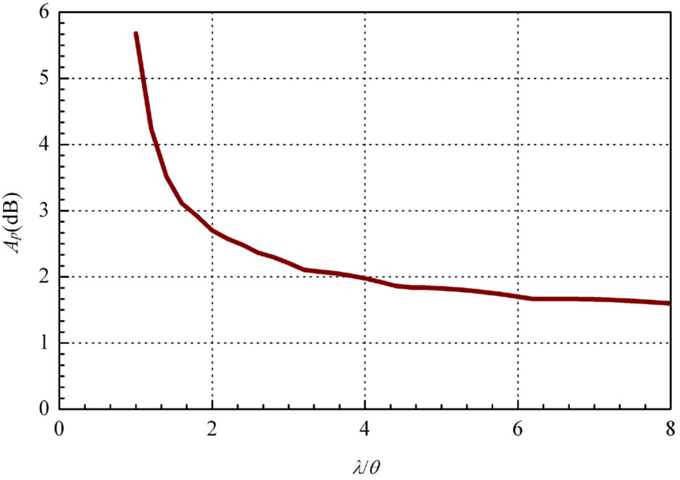


Figure: Resonance peak of the H_2 control system with an integrating plant

Design for the Second-Order Plant

By employing the first-order Taylor series expansion, the approximate plant is obtained as follows:

$$G(s) = \frac{K(1-\theta s)}{s(\tau s+1)}$$

With a similar design procedure, the optimal controller is

$$Q_{opt}(s) = \frac{s(\tau s+1)}{K(1+\theta s)}$$

The filter is taken as

$$J(s) = \frac{(2\lambda+2\theta)s+1}{(\lambda s+1)^2}$$

It follows that

$$C(s) = \frac{1}{K}\frac{(\tau s+1)[(2\lambda+2\theta)s+1]}{\lambda^2\theta s^2+(\lambda^2+4\lambda\theta+2\theta^2)s}$$

For the PID controller of the form

$$C(s) = K_C\left(1+\frac{1}{T_I s}+T_D s\right)\frac{1}{T_F s+1}$$

parameters are as follows:

$$T_F = \frac{\lambda^2\theta}{\lambda^2+4\lambda\theta+2\theta^2}, \qquad T_I = 2\lambda+2\theta+\tau$$

$$T_D = \frac{(2\lambda+2\theta)\tau}{T_I}, \qquad K_C = \frac{1}{K}\frac{T_I}{\lambda^2+4\lambda\theta+2\theta^2}$$

Example

Consider a distillation column. The materials to be separated are a mixture of three isomers and a small amount of other heavy components. To increase the production rate, the distillation column is designed such that there is very little excess separation ability. This makes tight control very important.

The control strategy is illustrated in Figure. The objective is to keep the distillate composition nearly pure in the lightest isomer while maintaining it at a very low level in the tails stream. The heat to the column is fixed, because the heat source is a vapor boiler that runs best at a fixed rate. The feed is set to fix the overall production rate for this part of the process. The base level is controlled by manipulating the tails flow rate. The overhead condensate tank level is controlled by manipulating the reflux flow. Under these conditions, composition control can be accomplished by controlling the middle column temperature.

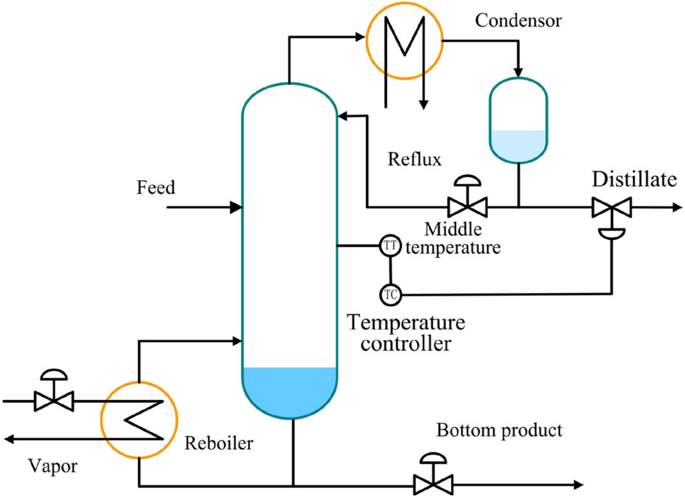


Figure: Control system of an high-purity distillation

Example (ctd.1)

Through an open-loop step test, the process model was developed:

$$\boxed{G(s) = \frac{0.01}{s} e^{-5.5s}}$$

Take $\lambda = 25$ for the H_2 PID controller:

$$\boxed{C(s) = \frac{100(138.75s^2 + 58s + 1)}{s(2.75\lambda^2 s + \lambda^2 + 11\lambda + 15.125)}}$$

and $\lambda = 16$ for the H_∞ PID controller:

$$\boxed{C(s) = \frac{100[(8.25\lambda + 7.5625)s^2 + (3\lambda + 5.5)s + 1]}{s(\lambda^3 s + 3\lambda^2 + 8.25\lambda + 7.5625)}}$$

Example (ctd.2)

A unit step reference is added at $t = 0$ and a unit step load is added at $t = 200$. The nominal responses of the closed-loop system are shown in Figure. It is observed that both of the two controllers give large overshoots and long settling times. This is an evident feature for the system with an integrating plant

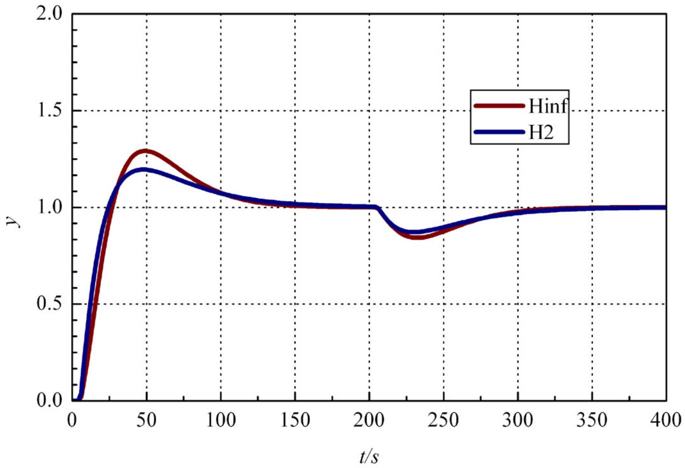


Figure: Nominal responses of H_2 PID and H_∞ PID

7.4 Controller Design for General Integrating Plants

Implementation: The IMC structure must be **abandoned** for the implementation of control systems, since there always exist zero-pole cancellations at the origin, which will cause the internal instability problem

Implementation: The IMC controller $Q(s)$ could **be utilized** to design the unity feedback loop controller

Design for general plants: To simplify the presentation, only the integrating plant with a simple pole at the origin is considered

Somple poles: A pole is simple if its multiplicity is one

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Design for general plants: To simplify the presentation, only the integrating plant with a simple pole at the origin is considered

Somple poles: A pole is simple if its multiplicity is one

Assume that the integrating plant is expressed by

$$G(s) = \frac{KN_+(s)N_-(s)}{sM_-(s)}e^{-\theta s}$$

where K is the gain
θ is the time delay
$N_-(s)$ and $M_-(s)$ are the polynomials with roots in the LHP
$N_+(s)$ is a polynomial with roots in the RHP
$N_+(0) = N_-(0) = M_-(0) = 1$
$\deg\{N_+(s)\} + \deg\{N_-(s)\} \leq \deg\{M_-(s)\} + 1$

Quasi-H$_\infty$ control:

By following the quasi-H$_\infty$ controller design procedure for stable plants, the desired closed-loop transfer function is chosen as

$$T(s) = N_+(s)J(s)e^{-\theta s}$$

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By following the quasi-H$_\infty$ controller design procedure for stable plants, the desired closed-loop transfer function is chosen as

$$T(s) = N_+(s)J(s)e^{-\theta s}$$

where $J(s)$ is a filter

$$J(s) = \frac{(\beta s + 1)}{(\lambda s + 1)^{n_j}}$$

λ is the performance degree,

$$n_j = \begin{cases} 2 + \deg\{M_-\} - \deg\{N_-\} & \deg\{M_-\} + 1 > \deg\{N_-\} \\ 2 & \deg\{M_-\} + 1 = \deg\{N_-\} \end{cases}$$

and β is determined by the following constraints:

$$\lim_{s \to 0} \frac{d^k}{ds^k}[1 - T(s)] = 0, k = 0, 1$$

or equivalently,

$$\begin{aligned} \lim_{s \to 0}[1 - N_+(s)J(s)e^{-\theta s}] &= 0 \\ \lim_{s \to 0} \frac{d}{ds}[1 - N_+(s)J(s)e^{-\theta s}] &= 0 \end{aligned}$$

Notice that one zero is introduced to $T(s)$

Since both $G(s)$ and $T(s)$ are known, the controller can be analytically derived:

$$Q(s) = \frac{T(s)}{G(s)} = \frac{1}{K}\frac{sM_{-}(s)(\beta s+1)}{N_{-}(s)(\lambda s+1)^{n_j}}$$

The unity feedback loop controller is

$$\begin{aligned} C(s) &= \frac{T(s)}{1-T(s)}\frac{1}{G(s)} \\ &= \frac{1}{K}\frac{sM_{-}(s)(\beta s+1)}{N_{-}(s)[(\lambda s+1)^{n_j}-(\beta s+1)N_{+}(s)e^{-\theta s}]} \end{aligned}$$

The reason the IMC structure cannot be used to control integrating plants is that there are zero-pole cancellations at the origin between $Q(s)$ and $G(s)$

One may want to solve the problem by combining $Q(s)$ and $G(s)$ into $C(s)$. Unfortunately, the problem cannot be overcome so easily. It can be verified that

$$\lim_{s\to 0}[(\lambda s+1)^{n_j}-(\beta s+1)N_+(s)e^{-\theta s}]=0$$

which implies that the denominator of $C(s)$ has a root at the origin. Since there is a time delay in the denominator, the root cannot be directly removed. As a result, the obtained $C(s)$ cannot guarantee the internal stability of the closed-loop system

Only after the RHP root is removed by employing rational approximations, can $C(s)$ guarantee the internal stability of the closed-loop system

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The design for quasi-H_∞ controllers can also be carried out as follows:

1. If the plant does not have a time delay, turn to 3.
2. If the plant contains a time delay, take the rational part of the plant as the nominal plant.
3. If the nominal plant does not have any zeros in the RHP, take its inverse as $Q_{opt}(s)$ and turn to 5.
4. If the nominal plant has zeros in the RHP, remove the factor that contains these zeros and take the inverse of the remainder as $Q_{opt}(s)$.
5. Introduce a filter to $Q_{opt}(s)$, compute the controller $C(s)$ and remove the RHP zero-pole cancellation in $C(s)$.

H_2 control:

The integrating plant is

$$G(s) = \frac{KN_+(s)N_-(s)}{sM_-(s)} e^{(-\theta s)}$$

The performance index is min $\|W(s)S(s)\|_2$. The internal stability imposes a constraint on $Q(s)$:

$$\lim_{s \to 0} [1 - G(s)Q(s)] = 0$$

The $Q(s)$ that satisfies the constraint has the following expression:

$$Q(s) = \frac{s[1 + sQ_1(s)]}{K}$$

where $Q_1(s)$ is stable

Then

$$
\begin{aligned}
& \|W(s)S(s)\|_2^2 \\
= & \left\| \frac{1}{s} \left\{ 1 - \frac{N_+(s)N_-(s)}{M_-(s)} e^{-\theta s} [1 + sQ_1(s)] \right\} \right\|_2^2 \\
= & \left\| \frac{1}{s} \left\{ \frac{N_+(-s)}{N_+(s)} e^{\theta s} - \frac{N_+(-s)N_-(s)}{M_-(s)} [1 + sQ_1(s)] \right\} \right\|_2^2 \\
= & \left\| \frac{e^{\theta s} N_+(-s) - N_+(s)}{sN_+(s)} \right\|_2^2 + \\
& \left\| \frac{1}{s} - \frac{N_+(-s)N_-(s)}{sM_-(s)} [1 + sQ_1(s)] \right\|_2^2
\end{aligned}
$$

Solving the optimal problem yields

$$
Q_{1opt}(s) = \frac{M_-(s) - N_+(-s)N_-(s)}{sN_+(-s)N_-(s)}
$$

Hence

$$Q_{opt}(s) = \frac{sM_-(s)}{KN_+(-s)N_-(s)}$$

Introduce the following filter:

$$J(s) = \frac{\beta s + 1}{(\lambda s + 1)^{n_j}}$$

where λ is the performance degree,

$$n_j = \begin{cases} 2 + \deg\{M_-\} - \{N_+\} - \{N_-\} & \{M_-\} + 1 > \{N_+\} + \{N_-\} \\ 2 & \{M_-\} + 1 = \{N_+\} + \{N_-\} \end{cases}$$

Consequently,

$$Q(s) = Q_{opt}(s)J(s) = \frac{sM_-(s)(\beta s + 1)}{KN_+(-s)N_-(s)(s + 1)^{n_j}}$$

β is determined by the constraints for asymptotic tracking:

$$\lim_{s\to 0}\frac{d^k}{ds^k}[1 - G(s)Q(s)] = 0, k = 0, 1$$

or equivalently,

$$\begin{aligned}\lim_{s\to 0}[1 - \frac{N_+(s)}{N_+(-s)}J(s)e^{-\theta s}] &= 0\\ \lim_{s\to 0}\frac{d}{ds}[1 - \frac{N_+(s)}{N_+(-s)}J(s)e^{-\theta s}] &= 0\end{aligned}$$

The design procedure for H_2 controllers can be described in a similar way to that for quasi-H_∞ controllers, except Step 4:

4. When the nominal plant has zeros in the RHP, construct an all-pass transfer function by using the factor that contains these zeros and then remove the all-pass transfer function, take the inverse of the remainder as $Q_{opt}(s)$

Idea for design in this section: A Type 2 controller was derived by modifying a Type 1 controller

Question: What difference exists between the two controllers?

Design a Type 2 system: The first step is to determine the form of the $Q(s)$ for a Type 2 system. $Q(s)$ should satisfy the following two conditions:

$$\lim_{s\to 0}[1-G(s)Q(s)] = 0$$

$$\lim_{s\to 0}\frac{d}{ds}[1-G(s)Q(s)] = 0$$

The $Q(s)$ that satisfies the first condition can be expressed as

$$Q(s)=\frac{s[1+sQ_1(s)]}{K}$$

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$$\begin{aligned}\lim_{s\to 0}[1-G(s)Q(s)] &= 0\\ \lim_{s\to 0}\frac{d}{ds}[1-G(s)Q(s)] &= 0\end{aligned}$$

The $Q(s)$ that satisfies the first condition can be expressed as

$$Q(s)=\frac{s[1+sQ_1(s)]}{K}$$

where $Q_1(s)$ is stable. Substituting this into the left-hand side of the second condition, we have

$$\begin{aligned}
&\lim_{s\to 0}\frac{d}{ds}\left\{1-\frac{N_+(s)N_-(s)}{M_-(s)}e^{-\theta s}[1+sQ_1(s)]\right\} \\
= & -\lim_{s\to 0}\frac{d}{ds}\left[\frac{N_+(s)N_-(s)}{M_-(s)}e^{-\theta s}+s\frac{N_+(s)N_-(s)}{M_-(s)}e^{-\theta s}Q_1(s)\right]
\end{aligned}$$

The second condition gives

$$Q_1(0)=\theta+\frac{d}{ds}M_-(0)-\frac{d}{ds}N_+(0)-\frac{d}{ds}N_-(0)$$

Then the $Q(s)$ that satisfies the two conditions can be written as

$$Q(s)=\frac{s[1+sQ_1(0)+s^2Q_2(s)]}{K}$$

where $Q_2(s)$ is stable

Therefore,

$$
\begin{aligned}
&\|W(s)S(s)\|_2^2\\
=&\ \left\|\frac{1}{s^2}\left\{1-\frac{N_+(s)N_-(s)}{M_-(s)}e^{-\theta s}[1+sQ_1(0)+s^2Q_2(s)]\right\}\right\|_2^2\\
=&\ \left\|\frac{1}{s^2}\left\{\frac{N_+(-s)}{N_+(s)}e^{\theta s}-\frac{N_+(-s)N_-(s)}{M_-(s)}[1+sQ_1(0)+s^2Q_2(s)]\right\}\right\|_2^2\\
=&\ \left\|\frac{e^{\theta s}N_+(-s)-N_+(s)[1+\theta s-2\frac{d}{ds}N_+(0)s]}{s^2N_+(s)}\right\|_2^2+\\
&\left\|\frac{1+\theta s-2\frac{d}{ds}N_+(0)s}{s^2}-\frac{N_+(-s)N_-(s)}{s^2M_-(s)}[1+sQ_1(0)+s^2Q_2(s)]\right\|_2^2
\end{aligned}
$$

It is evident that when

$$Q_{2opt}(s) = \frac{M_-(s)[1+\theta s - 2\frac{d}{ds}N_+(0)s]M_-(s)}{s^2 N_+(-s)N_-(s)} - \frac{1+Q_1(0)s}{s^2}$$

the right-hand side is minimum. The optimal controller is

$$Q_{opt}(s) = \frac{sM_-(s)[1+\theta s - 2\frac{d}{ds}N_+(0)s]}{KN_+(-s)N_-(s)}$$

Introduce a Type 2 filter. The suboptimal controller is

$$Q(s) = \frac{sM_-(s)[1+\theta s - 2\frac{d}{ds}N_+(0)s]\{[n_j\lambda + 2\frac{d}{ds}N_+(0)]s+1\}}{KN_+(-s)N_-(s)(\lambda s+1)^{n_j}}$$

where

$$n_j = \begin{cases} 3+\{M_-\}-\{N_+\}-\{N_-\} & \{M_-\}+1 > \{N_+\}+\{N_-\} \\ 3 & \{M_-\}+1 = \{N_+\}+\{N_-\} \end{cases}$$

It is easy to verify that the optimal performance of the Type 2 controller designed for $W(s) = 1/s$ is

$$\min \|W(s)S(s)\|_2 = \left\| \frac{N_+(-s) - N_+(s)e^{-\theta s}}{sN_+(-s)} \right\|_2$$

while the optimal performance of the Type 2 controller designed for $W(s) = 1/s^2$ is

$$\min \|W(s)S(s)\|_2 = \left\| \frac{N_+(-s) - N_+(s)[1 + \theta s - 2\frac{d}{ds}N_+(0)s][2\frac{d}{ds}N_+(0)s + 1]e^{-\theta s}}{s^2 N_+(-s)} \right\|_2$$

The controller can also be design for quantitative responses:

Increase the performance degree monotonically until the required response is obtained

7.5 Maclaurin PID Controllers for Integrating Plants

Assume that the integrating plant has m poles at the origin:

$$G(s) = \frac{KN_+(s)N_-(s)}{s^m M_-(s)} e^{-\theta s}$$

Quasi-H$_\infty$ control: The closed-loop transfer function is

$$T(s) = N_+(s)J(s)e^{-\theta s}$$

where $J(s)$ is a filter:

$$J(s) = \frac{N_x(s)}{(\lambda s + 1)^{n_j}}$$

λ is the performance degree,

$$n_j = \begin{cases} 2m + \{M_-\} - \{N_-\} & \{M_-\} + m > \{N_-\} \\ m + 1 & \{M_-\} + m = \{N_-\} \end{cases}$$

$N_x(s)$ is a polynomial with its roots in the LHP, $N_x(0) = 1$, and $\deg\{N_x(s)\} = m$. $N_x(s)$ is determined by the asymptotic tracking constraints:

$$\lim_{s \to 0} \frac{d^k}{ds^k}[1 - T(s)] = 0, k = 0, 1, ..., m$$

Quasi-H$_2$ control: The closed-loop transfer function is

$$T(s) = \frac{N_+(s)}{N_+(-s)} J(s) e^{-\theta s}$$

where

$$J(s) = \frac{N_x(s)}{(\lambda s + 1)^{n_j}}$$

The determination of $N_x(s)$ is similar to that for the quasi-H$_\infty$ control

$$n_j = \begin{cases} 2m + \{M_-\} - \{N_+\} - \{N_-\} & \{M_-\} + m > \{N_+\} + \{N_-\} \\ m+1 & \{M_-\} + m = \{N_+\} + \{N_-\} \end{cases}$$

PID design: Both the quasi-H$_\infty$ controller and the H$_2$ controller can be computed by

$$C(s) = \frac{1}{G(s)} \frac{T(s)}{1 - T(s)}$$

Since $T(0) = 1$, $C(s)$ has a pole at the origin. Write $C(s)$ as

$$C(s) = \frac{f(s)}{s}$$

The Maclaurin series expansion of $C(s)$ is

$$C(s) = \frac{1}{s}\left[f(0) + f'(0)s + \frac{f''(0)}{2!}s^2 + \ldots \right]$$

Omit high-order terms. Only the first three terms are taken to approximate the ideal controller:

$$C(s) = K_C \left(1 + \frac{1}{T_I s} + T_D s\right)$$

whose parameters are

$$K_C = f'(0), \quad T_I = \frac{f'(0)}{f(0)}, \quad T_D = \frac{f''(0)}{2f'(0)}$$

Furthermore, define

$$f(s) = \frac{N(s)}{M(s)}$$

where $M(s)$ and $N(s)$ are polynomials

The values of $f(s)$ and its derivatives at the origin are

$$
\begin{aligned}
f(0) &= \frac{N(0)}{M(0)} \\
f'(0) &= \frac{N'(0)M(0) - M'(0)N(0)}{M(0)^2} \\
f''(0) &= \frac{N''(0)M(0)^2 - M''(0)N(0)M(0)}{M(0)^3} - \\
&\quad \frac{2M'(0)N'(0)M(0) + 2M'(0)^2N(0)}{M(0)^3}
\end{aligned}
$$

Two cases are considered: The plant is of first-order and the plant is of second-order. First, assume that the plant is

$$G(s) = \frac{K}{s}e^{-\theta s}$$

The closed-loop transfer function with the asymptotic tracking property is

$$T(s) = \frac{(2\lambda+\theta)s+1}{(\lambda s+1)^2}e^{-\theta s}$$

Then

$$\begin{aligned} N(s) &= \frac{(2\lambda+\theta)s+1}{K} \\ M(s) &= \frac{(\lambda s+1)^2 - [(2\lambda+\theta)s+1]e^{-\theta s}}{s^2} \end{aligned}$$

which yields

$$N(0) = \frac{1}{K}, \qquad N'(0) = \frac{2\lambda+\theta}{K}$$

$$N''(0) = 0, \qquad M(0) = \frac{2\lambda^2+4\lambda\theta+\theta^2}{2}$$

$$M'(0) = \frac{-3\lambda\theta^2-\theta^3}{3}, \qquad M''(0) = \frac{3\theta^4+8\theta^3\lambda}{12}$$

$f(s)$ and its first order and second order derivatives are

$$
\begin{aligned}
f(0) &= \frac{2}{K(2\lambda^2+4\lambda\theta+\theta^2)} \\
f'(0) &= \frac{2(12\lambda^3+30\lambda^2\theta+24\lambda\theta^2+5\theta^3)}{3K(2\lambda^2+4\lambda\theta+\theta^2)^2} \\
f''(0) &= \frac{\theta^2(288\lambda^4+768\lambda^3\theta+702\lambda^2\theta^2+252\lambda\theta^3+31\theta^4)}{9K(2\lambda^2+4\lambda\theta+\theta^2)^3}
\end{aligned}
$$

Accordingly, controller parameters are

$$
\begin{aligned}
T_I &= 2\lambda+\theta+\frac{2\theta^3+6\lambda\theta^2}{3(2\lambda^2+4\lambda\theta+\theta^2)} \\
K_C &= \frac{2T_I}{K(2\lambda^2+4\lambda\theta+\theta^2)} \\
T_D &= \frac{\theta^2(288\lambda^4+768\lambda^3\theta+702\lambda^2\theta^2+252\lambda\theta^3+31\theta^4)}{36T_I(2\lambda^2+4\lambda\theta+\theta^2)^2}
\end{aligned}
$$

The formula seems a bit complicated. However, the computation is not difficult, since all parameters of the plant are known.

Consider the second-order plant:

$$G(s) = \frac{K}{s(\tau s + 1)} e^{-\theta s}$$

The closed-loop transfer function with the asymptotic tracking property can be written as

$$T(s) = \frac{(3\lambda + \theta)s + 1}{(\lambda s + 1)^3} e^{-\theta s}$$

Then

$$\begin{aligned} N(s) &= \frac{(\tau s + 1)[(3\lambda + \theta)s + 1]}{K}, \\ M(s) &= \frac{(\lambda s + 1)^3 - [(3\lambda + \theta)s + 1]e^{-\theta s}}{s^2} \end{aligned}$$

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Then

$$\begin{aligned} N(s) &= \frac{(\tau s + 1)[(3\lambda + \theta)s + 1]}{K}, \\ M(s) &= \frac{(\lambda s + 1)^3 - [(3\lambda + \theta)s + 1]e^{-\theta s}}{s^2} \end{aligned}$$

This yields

$$N(0)=\frac{1}{K},\qquad N'(0)=\frac{3\lambda+\tau+\theta}{K}$$
$$N''(0)=\frac{2\tau(3\lambda+\theta)}{K},\qquad M(0)=\frac{6\lambda^2+6\lambda\theta+\theta^2}{2}$$
$$M'(0)=\frac{6\lambda^3-9\lambda\theta^2-2\theta^3}{6},\qquad M''(0)=\frac{\theta^4+4\lambda\theta^3}{4}$$

The values of $f(s)$ and its first order and second order derivatives at the origin are

$$f(0)=\frac{2}{K(6\lambda^2+6\lambda\theta+\theta^2)}$$
$$f'(0)=\frac{2(18\tau\lambda^2+3\theta^2\tau+18\tau\lambda\theta+72\lambda^2\theta+5\theta^3+36\lambda\theta^2+48\lambda^3)}{3K(6\lambda^2+\theta^2+6\theta\lambda)^2}$$

$$f''(0) = \frac{\begin{array}{l}8352\tau\lambda^3\theta^2 + 31\theta^6 - 1152\lambda^6 + 3456\tau\lambda^5 + 60\tau\theta^5 + \\ 1602\lambda^2\theta^4 + 378\lambda\theta^5 + 2640\lambda^3\theta^3 + 864\lambda^4\theta^2 - \\ 1728\lambda^5\theta + 8640\tau\lambda^4\theta + 792\tau\lambda\theta^4 + 3816\tau\lambda^2\theta^3\end{array}}{9K(6\lambda^2 + 6\lambda\theta + \theta^2)^3}$$

Therefore, controller parameters are

$$T_I = \tau + \frac{5\theta^3 + 36\lambda\theta^2 + 48\lambda^3 + 72\theta\lambda^2}{3(6\lambda^2 + 6\lambda\theta + \theta^2)}$$

$$K_C = \frac{2T_I}{K(6\lambda^2 + 6\lambda\theta + \theta^2)}$$

$$T_D = \frac{\begin{array}{l}240\tau\lambda^4\theta + 22\tau\lambda\theta^4 + 232\tau\lambda^3\theta^2 + 106\tau\lambda^2\theta^3 + \\ 5\tau\theta^5/3 + 96\tau\lambda^5 + 89\theta^4\lambda^2/2 + 31\theta^6/36 - 48\lambda^5\theta + \\ 24\lambda^4\theta^2 + 21\lambda\theta^5/2 + 220\lambda^3\theta^3/3 - 32\lambda^6\end{array}}{T_I(6\lambda^2 + 6\lambda\theta + \theta^2)^2}$$

7.6 Best Achievable Performance of a PID Controllers

Compromise in designing a PID controller:

Good performance<=> Complicated form

Not so good a result <=> Simple form

PID with best achievable performance: A general integrating plant can be described as

$$G(s) = \frac{KN_+(s)N_-(s)}{s^m M_-(s)} e^{-\theta s}$$

Suppose that the closed-loop transfer function $T(s)$is given as that in (1) or (1). The controller can be computed by

$$C(s) = \frac{1}{G(s)} \frac{T(s)}{1 - T(s)}$$

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Suppose that the closed-loop transfer function $T(s)$is given as that in (1) or (1). The controller can be computed by

$$C(s) = \frac{1}{G(s)} \frac{T(s)}{1 - T(s)}$$

$C(s)$ has a pole at the origin. Express it as

$$C(s) = \frac{f(s)}{s}$$

Using the Maclaurin series expansion, we have

$$f(s) = f(0) + f'(0)s + \frac{f''(0)}{2!}s^2 + \frac{f^{(3)}(0)}{3!}s^3 + \ldots$$

A practical PID controller can be written as

$$C(s) = \frac{a_2 s^2 + a_1 s + a_0}{s(b_1 s + 1)}$$

Let the Pade approximation of $f(s)$ be

$$\frac{a_2 s^2 + a_1 s + a_0}{b_1 s + 1}$$

Then

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} f(0) & 0 \\ f'(0) & f(0) \\ f''(0)/2! & f'(0) \end{bmatrix} \begin{bmatrix} 1 \\ b_1 \end{bmatrix}$$

$$b_1 f''(0)/2! = -f^{(3)}(0)/3!$$

This yields

$$a_0 = f(0), \qquad a_1 = b_1 f(0) + f'(0)$$

$$a_2 = b_1 f'(0) + f''(0)/2!, \qquad b_1 = -\frac{f^{(3)}(0)}{3f''(0)}$$

If the PID controller is in the form of

$$C = K_C\left(1 + \frac{1}{T_I s} + T_D s\right)\frac{1}{T_F s + 1}$$

A little computations give

$$K_C = a_1, \quad T_I = \frac{a_1}{a_0}, \quad T_D = \frac{a_2}{a_1}, \quad T_F = b_1$$

As discussed in Section 5.6, all of these parameters should be chosen as positive numbers

Consider the plant

$$G(s) = \frac{Ke^{-\theta s}}{s}$$

The value of $f(s)$ and its derivatives at the origin are

$$\begin{aligned}
f(0) &= \frac{2}{K(2\lambda^2 + \theta^2 + 4\lambda\theta)} \\
f'(0) &= \frac{2(12\lambda^3 + 24\lambda\theta^2 + 30\lambda^2\theta + 5\theta^3)}{3K(2\lambda^2 + \theta^2 + 4\lambda\theta)^2} \\
f''(0) &= \frac{\theta^2(768\theta\lambda^3 + 252\lambda\theta^3 + 702\theta^2\lambda^2 + 31\theta^4 + 288\lambda^4)}{9K(2\lambda^2 + \theta^2 + 4\lambda\theta)^3}
\end{aligned}$$

$$f^{(3)}(0) = \frac{\theta^3(121\theta^6 - 2880\lambda^6 + 1044\theta^2\lambda^4 + 5040\theta\lambda^5 + 1248\lambda\theta^5 + 4620\lambda^2\theta^4 + 6696\lambda^3\theta^3)}{45K(2\lambda^2 + \theta^2 + 4\lambda\theta)^4}$$

Controller parameters are obtained as follows:

$$a_0 = \frac{2}{K(2\lambda^2 + \theta^2 + 4\lambda\theta)}$$

$$a_1 = \frac{4(109\theta^5 + 1026\lambda\theta^4 + 3648\lambda^2\theta^3 + 6090\lambda^3\theta^2 + 4800\theta\lambda^4 + 1440\lambda^5)}{5K(252\theta^3\lambda + 702\theta^2\lambda^2 + 768\theta\lambda^3 + 31\theta^4 + 288\lambda^4)(2\lambda^2 + \theta^2 + 4\lambda\theta)}$$

$$a_2 = \frac{(265\theta^5 + 2496\lambda\theta^4 + 9000\lambda^2\theta^3 + 15408\theta^2\lambda^3 + 12480\theta\lambda^4 + 3840\lambda^5)}{10K(768\theta\lambda^3 + 252\theta^3\lambda + 702\theta^2\lambda^2 + 31\theta^4 + 288\lambda^4)(2\lambda^2 + \theta^2 + 4\lambda\theta)}$$

$$b_1 = -\frac{\theta(121\theta^6 + 1248\theta^5\lambda - 2880\lambda^6 + 6696\theta^3\lambda^3 - 5040\theta\lambda^5 + 1044\theta^2\lambda^4 + 4620\theta^4\lambda^2)}{15(768\theta\lambda^3 + 252\theta^3\lambda + 702\theta^2\lambda^2 + 31\theta^4 + 288\lambda^4)(2\lambda^2 + \theta^2 + 4\lambda\theta)}$$

End of Chapter 7