Chapter 5 H_2 PID Controllers for Stable Plants

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5.1 H_2 PID Controllers for the First-Order Plant

An analog of H_∞ optimal control theory: H_2 optimal control theory

Assume that the plant is

$$G(s) = \frac{Ke^{-\theta s}}{\tau s + 1}$$

Using the Youla parameterization, we have

$$C(s) = \frac{Q(s)}{1 - G(s)Q(s)}$$

where $Q(s)$ is a stable transfer function. It is difficult to treat $e^{-\theta s}$ analytically. Approximate it by the 1/1 Pade approximant:

$$G(s) \approx K\frac{1 - \theta s/2}{(\tau s + 1)(1 + \theta s/2)}$$

The design procedure for the H_2 PID controller is similar to that for the H_∞ PID controller. The controller is first designed for the **approximate plant** and then used to control the **original plant**

The H_2 optimal index is

$$\min \|W(s)S(s)\|_2$$

where $W(s)$ is the weighting function. Assume that the system input is a unit step. In view of the discussion in Section 3.2, the weighting function in H_2 optimal control should be chosen so that the input is normalized to an impulse, that is, $\|d(s)/W(s)\|_2 = 1$. Then, $W(s) = 1/s$.

$W(s)$ has a pole on the imaginary axis. To guarantee a finite 2-norm and to have the asymptotic property, a constraint has to be imposed on the design:

$$\lim_{s\to 0} S(s) = \lim_{s\to 0}[1 - G(s)Q(s)] = 0$$

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$$\lim_{s\to 0} S(s) = \lim_{s\to 0}[1 - G(s)Q(s)] = 0$$

In other words, $S(s)$ must have a zero at the origin to cancel the pole of $W(s)$. This gives

$$Q(0) = \frac{1}{G(0)} = \frac{1}{K}$$

It should be emphasized that the constraint is also required for asymptotic tracking. The set of all $Q(s)$s satisfying the constraint can be written as

$$Q(s) = \frac{1}{K} + sQ_1(s)$$

where $Q_1(s)$ is stable. The function to be minimized is

$$\begin{aligned} & \|W(s)S(s)\|_2^2 \\ = & \left\| W(s)\left\{1 - G(s)\left[\frac{1}{K} + sQ_1(s)\right]\right\}\right\|_2^2 \\ = & \left\| \frac{\theta\tau s/2 + (\theta+\tau)}{(\tau s+1)(\theta s/2+1)} - \frac{K(1-\theta s/2)}{(\tau s+1)(1+\theta s/2)} Q_1(s)\right\|_2^2 \end{aligned}$$

$$= \left\| \frac{1-\theta s/2}{1+\theta s/2} \left[\frac{\theta\tau s/2 + (\theta+\tau)}{(\tau s+1)(1-\theta s/2)} - \frac{K}{\tau s+1} Q_1(s) \right] \right\|_2^2$$

$(1-\theta s/2)/(1+\theta s/2)$ in the equation is an all-pass transfer function. With the definition of 2-norm, it is easy to verify that the 2-norm of a transfer function keeps its value after introducing an all-pass transfer function to it. Therefore,

$$\|W(s)S(s)\|_2^2 = \left\| \frac{\theta\tau s/2 + (\theta+\tau)}{(\tau s+1)(1-\theta s/2)} - \frac{K}{\tau s+1} Q_1(s) \right\|_2^2$$

As we known, by partial fraction expansion a strictly proper transfer function without poles on the imaginary axis can always be uniquely expressed as a stable part (which does not have poles in Re $s > 0$) and an unstable part (which does not have poles in Re $s < 0$):

$$\frac{\theta\tau s/2 + (\theta + \tau)}{(\tau s + 1)(1 - \theta s/2)} = \frac{\theta}{1 - \theta s/2} + \frac{\tau}{\tau s + 1}$$

Then

$$\|W(s)S(s)\|_2^2 = \left\|\frac{\theta}{1 - \theta s/2}\right\|_2^2 + \left\|\frac{\tau}{\tau s + 1} - \frac{K}{\tau s + 1}Q_1(s)\right\|_2^2$$

Temporarily relax the requirement on the properness of $Q(s)$. To obtain the minimum, the only choice is

$$Q_{1opt}(s) = \frac{\tau}{K}$$

Consequently, the optimal $Q(s)$ is

$$Q_{opt}(s) = \frac{\tau s + 1}{K}$$

$Q(s)$ should be proper. Use the following filter to roll the improper solution off:

$$J(s) = \frac{1}{\lambda s + 1}$$

where λ is the performance degree. It is a positive real number. The suboptimal $Q(s)$ is

$$Q(s) = Q_{opt}(s)J(s) = \frac{\tau s + 1}{K(\lambda s + 1)}$$

Since $Q(0) = 1/K$, $Q(s)$ satisfies the constraint for asymptotic tracking. The unity feedback loop controller is

$$C(s) = \frac{Q(s)}{1 - G(s)Q(s)} = \frac{1}{K}\frac{(\tau s + 1)(1 + \theta s/2)}{\theta\lambda s^2/2 + (\lambda + \theta)s}$$

Comparing the controller with

$$C = K_C\left(1 + \frac{1}{T_I s} + T_D s\right)\frac{1}{T_F s + 1}$$

gives that

$$T_F = \frac{\theta\lambda}{2(\lambda+\theta)}, T_I = \tau + \frac{\theta}{2}, T_D = \frac{\theta\tau}{2T_I}, K_C = \frac{T_I}{K(\lambda+\theta)}$$

If the following form is chosen:

$$C(s) = K_C\left(1 + \frac{1}{T_I s} + \frac{T_D s}{T_F s + 1}\right)$$

the parameters of the PID controller are

$$T_F = \frac{\theta\lambda}{2(\lambda+\theta)}, T_I = \tau + \frac{\theta}{2} - T_F, T_D = \frac{\theta\tau}{2T_I} - T_F, K_C = \frac{T_I}{K(\lambda+\theta)}$$

When the PID controller is in the form of

$$C(s) = K_C\left(1+\frac{1}{T_I s}\right)\frac{T_D s+1}{T_F s+1}$$

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$$T_F = \frac{\theta\lambda}{2(\lambda+\theta)}, T_I = \tau(or\frac{\theta}{2}), T_D = \frac{\theta}{2}(\text{or }\tau), K_C = \frac{T_I}{K(\lambda+\theta)}$$

The optimal performance, approximately, is

$$\min \|W(s)S(s)\|_2 = \left\|\frac{\theta}{1-\theta s/2}\right\|_2 = \sqrt{\theta}$$

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5.2 Quantitative Tuning of H_2 PID Controllers

Nominal stability: The larger the time delay, the more difficult to stabilize the closed-loop system

As long as the performance degree is greater than a lower bound, the closed-loop system is stable

Nominal performance: The existence of time delays adversely affects the performance of the closed-loop system. The performance is worse and worse with the increase of the time delay

The performance degree of the H_2 PID controller has a similar function to that of the H_∞ PID controller:

When there is no modeling error, the performance degree can be used to tune the response shape of the nominal closed-loop system quantitatively

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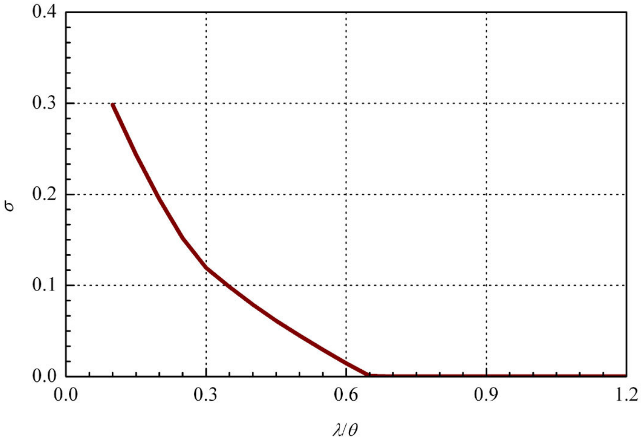

Figure: Relationship between the performance degree and the overshoot

e.g., 12% overshoot $->\lambda=0.3\theta$

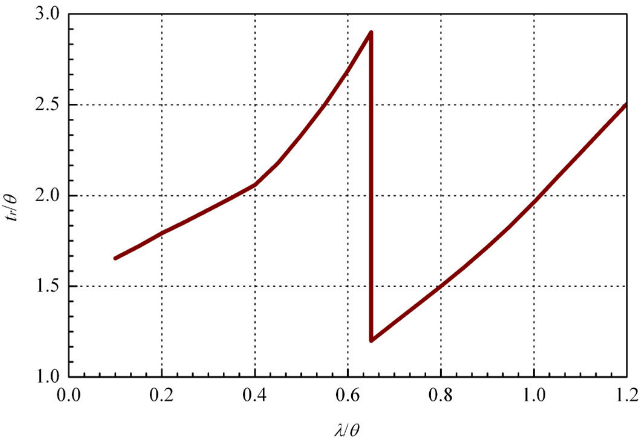

Figure: Relationship between the performance degree and the rise time

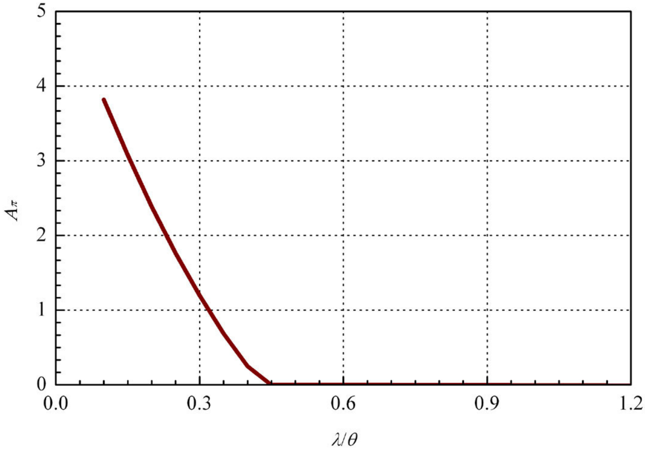

Figure: Relationship between the performance degree and the resonance peak

e.g., 2dB $->\lambda = 0.22\theta$

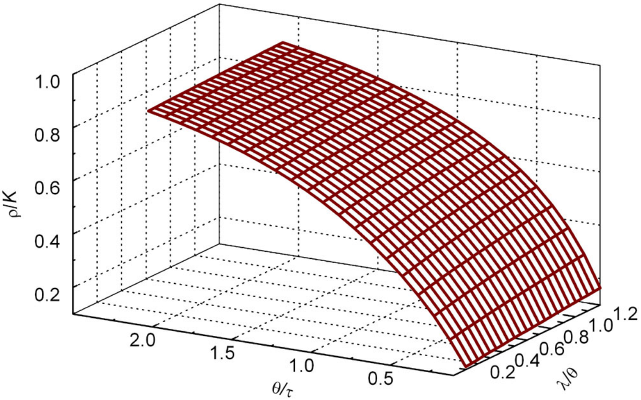

Figure: Relationship between the performance degree and the perturbation peak

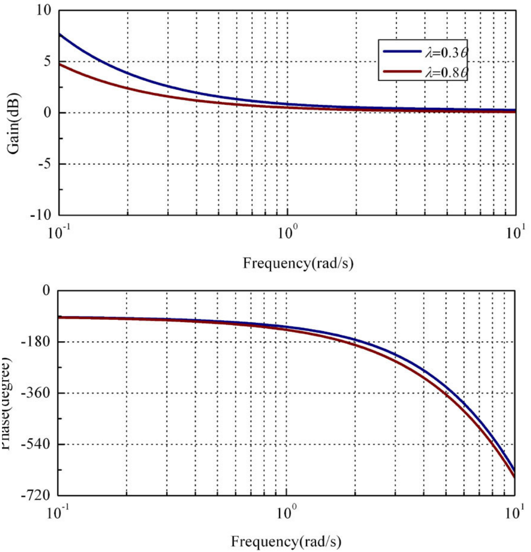

Figure: Bode plot of the H_2 PID control system

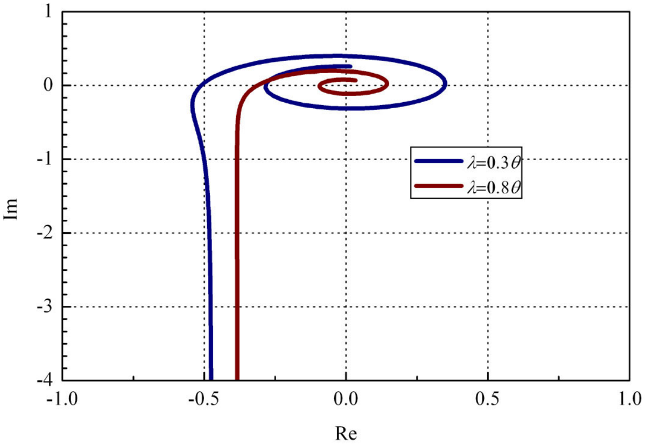

Figure: Nyquist plot of the H_2 PID control system

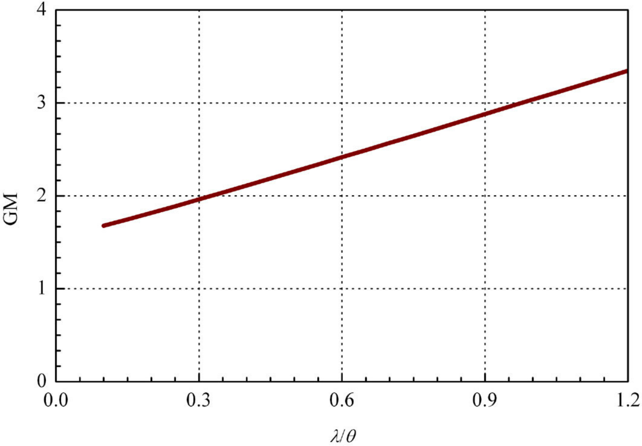

Figure: Relationship between the performance degree and the gain margin

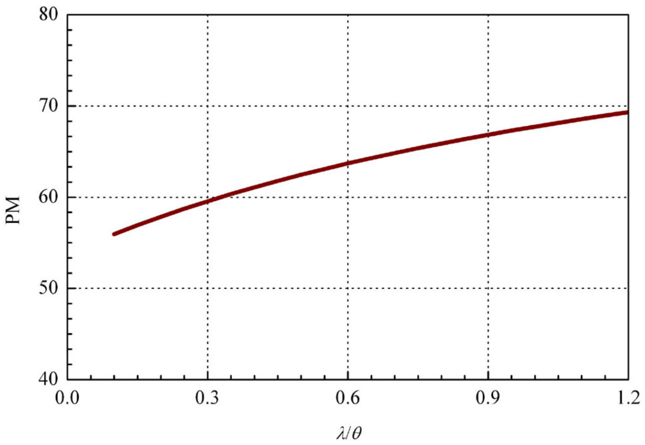

Figure: Relationship between the performance degree and the phase margin

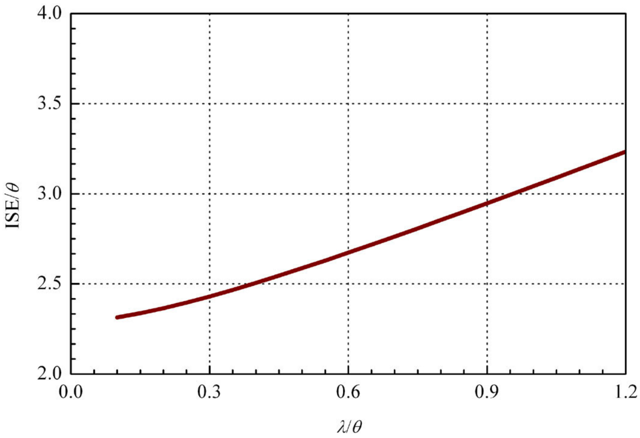

Figure: Relationship between the performance degree and ISE

Robust performance:

Increase the performance degree monotonically until the required response is obtained

The robust performance problem is not always solvable. Design methods in this book provide an easily checked solution to this problem.By adjusting the performance, it is easy to know whether or not the required overshoot is achievable for some uncertain plant

Implementation of PID controllers: In traditional PID controllers, T_F is fixed (usually $0.1T_D$). If a traditional PID controller has been installed in a system and one desires to use the tuning method here, then the T_F in the analytical formulas can be omitted and only the other three parameters are used for tuning. The responses are similar

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Example

Consider a strip thickness control system. A typical tandem hot strip mill is depicted in Figure. The metal slab is first heated to a certain temperature in the reheating furnace. Its thickness is then reduced in the roughing mill stand and finally refined in the finishing mill stand. At the exit, the strip is cooled and coiled by the down coiler. One main quantity to be controlled in the process is the thickness of the strip. The thickness is controlled through the roll force of the finishing mill.

It is known that the distance from the thickness meter to the finishing mill stand is 4.9m, the speed of the strip is 0.7m/s, and the time constant of the actuator is 3s. Then the transfer function of the plant can be written as

$$\boxed{G(s) = \frac{0.2e^{-7s}}{3s+1}}$$

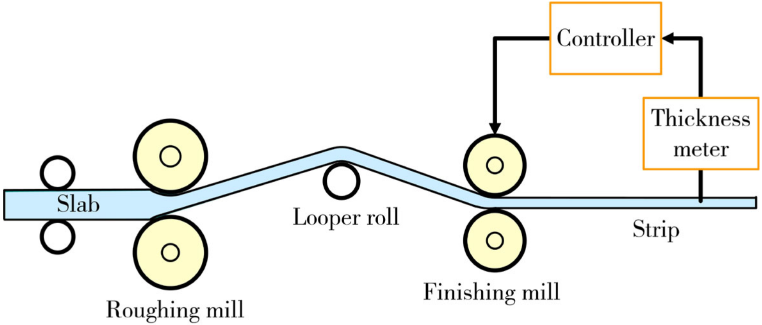

Figure: Control system for the thickness

Example (ctd.1)

In light of the design in the last section, the H_2 PID controller is

$$\boxed{C(s) = \frac{1}{0.2}\frac{(3s+1)(3.5s+1)}{3.5\lambda s^2 + (\lambda+7)s}.}$$

The performance degree is taken to be $\lambda = 0.3\theta$, which corresponds to about 12% overshoot according to Figure. A unit step reference is added at $t = 0$ and a unit step load is added at $t = 100$. The nominal response of the closed-loop system is shown in Figure. The controller provides good response for the plant with large time delay.

Now take $T_F = 0.1T_D$ in the H_2 PID controller. It is seen in Figure that the response given by the approximate H_2 PID controller is similar to that given by the original H_2 PID controller

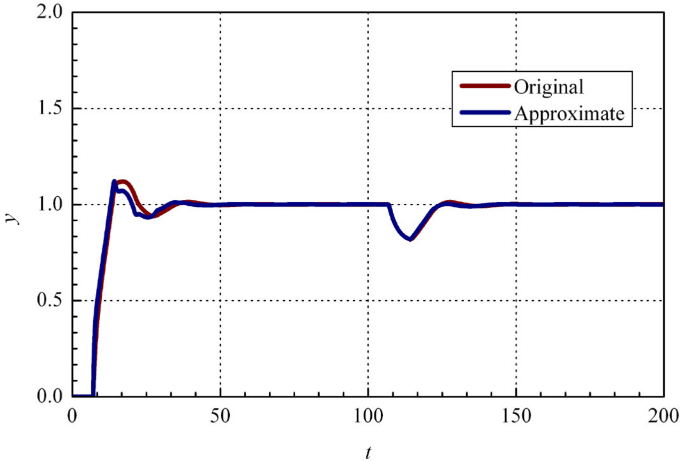

Figure: System response for the H_2 PID controller

5.3 H_2 PID Controllers for the Second-Order Plant

Assume that the plant is given by

$$G(s) = \frac{Ke^{-\theta s}}{(\tau_1 s + 1)(\tau_2 s + 1)}$$

By employing the first-order Taylor series expansion, the following approximate plant is obtained:

$$G(s) \approx \frac{K(1 - \theta s)}{(\tau_1 s + 1)(\tau_2 s + 1)}$$

Define the optimal performance index as

$$\min \|W(s)S(s)\|_2$$

If the system input is a unit step, $W(s) = 1/s$ is taken

To guarantee a finite 2-norm and to have the asymptotic property, the following constraint should be satisfied:

$$\lim_{s\to 0}[1 - G(s)Q(s)] = 0$$

It follows that

$$Q(0) = \frac{1}{G(0)} = \frac{1}{K}$$

Then all $Q(s)$s that satisfy the constraint are in the form of

$$Q(s) = \frac{1}{K} + sQ_1(s)$$

where $Q_1(s)$ is a stable transfer function. Then

$$\|W(s)S(s)\|_2^2 = \left\| W(s)\left\{1 - G(s)\left[\frac{1}{K} + sQ_1(s)\right]\right\}\right\|_2^2$$

$$
\begin{aligned}
&= \left\| \frac{\tau_1\tau_2 s + \tau_1 + \tau_2 + \theta}{(\tau_1 s+1)(\tau_2 s+1)} - \frac{K(1-\theta s)Q_1(s)}{(\tau_1 s+1)(\tau_2 s+1)} \right\|_2^2 \\
&= \left\| \frac{(\tau_1\tau_2 s + \tau_1 + \tau_2 + \theta)(1+\theta s)}{(\tau_1 s+1)(\tau_2 s+1)(1-\theta s)} - \frac{K(1+\theta s)Q_1(s)}{(\tau_1 s+1)(\tau_2 s+1)} \right\|_2^2 \\
&= \left\| \frac{2\theta}{1-\theta s} + \frac{(\tau_1\tau_2 s + \tau_1 + \tau_2 - \theta)}{(\tau_1 s+1)(\tau_2 s+1)} - \frac{K(1+\theta s)Q_1(s)}{(\tau_1 s+1)(\tau_2 s+1)} \right\|_2^2
\end{aligned}
$$

Expanding the right-hand side gives that

$$
\begin{aligned}
&\|W(s)S(s)\|_2^2 \\
= &\left\| \frac{2\theta}{1-\theta s} \right\|_2^2 + \left\| \frac{(\tau_1\tau_2 s + \tau_1 + \tau_2 - \theta)}{(\tau_1 s+1)(\tau_2 s+1)} - \frac{K(1+\theta s)Q_1(s)}{(\tau_1 s+1)(\tau_2 s+1)} \right\|_2^2
\end{aligned}
$$

Minimize $\|W(s)S(s)\|_2$. The unique optimal solution is

$$
Q_{1opt}(s) = \frac{\tau_1\tau_2 s + \tau_1 + \tau_2 - \theta}{K(1+\theta s)}
$$

$$
\begin{aligned}
&= \left\| \frac{\tau_1\tau_2 s + \tau_1 + \tau_2 + \theta}{(\tau_1 s + 1)(\tau_2 s + 1)} - \frac{K(1-\theta s)Q_1(s)}{(\tau_1 s + 1)(\tau_2 s + 1)} \right\|_2^2 \\
&= \left\| \frac{(\tau_1\tau_2 s + \tau_1 + \tau_2 + \theta)(1+\theta s)}{(\tau_1 s + 1)(\tau_2 s + 1)(1-\theta s)} - \frac{K(1+\theta s)Q_1(s)}{(\tau_1 s + 1)(\tau_2 s + 1)} \right\|_2^2 \\
&= \left\| \frac{2\theta}{1-\theta s} + \frac{(\tau_1\tau_2 s + \tau_1 + \tau_2 - \theta)}{(\tau_1 s + 1)(\tau_2 s + 1)} - \frac{K(1+\theta s)Q_1(s)}{(\tau_1 s + 1)(\tau_2 s + 1)} \right\|_2^2
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\end{aligned}
$$

Minimize $\|W(s)S(s)\|_2$. The unique optimal solution is

$$
Q_{1opt}(s) = \frac{\tau_1\tau_2 s + \tau_1 + \tau_2 - \theta}{K(1+\theta s)}
$$

Consequently,

$$Q_{opt}(s) = \frac{(\tau_1 s + 1)(\tau_2 s + 1)}{K(1 + \theta s)}$$

Introduce the following filter to roll $Q_{opt}(s)$ off at high frequencies:

$$J(s) = \frac{1}{\lambda s + 1}$$

We have

$$Q(s) = Q_{opt}(s)J(s) = \frac{(\tau_1 s + 1)(\tau_2 s + 1)}{K(1 + \theta s)(\lambda s + 1)}$$

$Q(s)$ satisfies the constraint for asymptotic tracking. The unity feedback loop controller is

$$C(s) = \frac{Q(s)}{1 - G(s)Q(s)} = \frac{1}{K}\frac{(\tau_1 s + 1)(\tau_2 s + 1)}{\lambda\theta s^2 + (\lambda + 2\theta)s}$$

Compare it with

$$C(s) = K_C \left(1 + \frac{1}{T_I s} + T_D s\right) \frac{1}{T_F s + 1}$$

parameters of the PID controller are

$$T_F = \frac{\lambda\theta}{2\lambda + \theta}, T_I = \tau_1 + \tau_2, T_D = \frac{\tau_1\tau_2}{\tau_1 + \tau_2}, K_C = \frac{\tau_1 + \tau_2}{K(\lambda + 2\theta)}$$

Normally, the value of λ is chosen between 0.2θ and 1.2θ

Example

Consider the plant given in the last chapter:

$$\boxed{G(s) = \frac{0.54e^{-15s}}{(15s + 1)^2}}$$

Example (ctd.1)

Take $\lambda = 0.9\theta$ for the H_∞ PID controller:

$$\boxed{C(s) = \frac{1}{0.54}\frac{(15s+1)^2}{\lambda^2 s^2 + (2\lambda + 15)s}}$$

The H_2 PID controller is

$$\boxed{C(s) = \frac{1}{0.54}\frac{(15s+1)^2}{15\lambda s^2 + (\lambda + 30)s}}$$

The parameter of the H_2 PID controller is chosen in such a way that the closed-loop system has the same overshoot as that with an H_∞ PID controller. In this case, $\lambda = 0.78\theta$. A unit step reference is added at $t = 0$ and a unit step load is added at $t = 300$. The nominal responses of the closed-loop system are shown in Figure. The two controllers provide similar responses

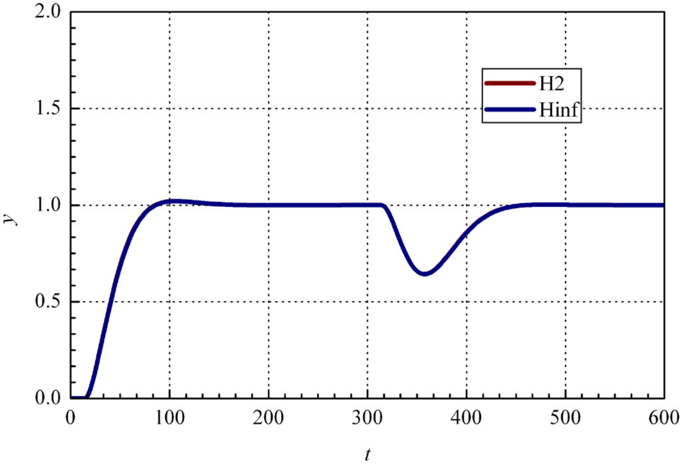

Figure: Responses of H_∞ and H_2 PID controllers

Load Responses

Note: The disturbance is always added at the plant input in simulations when the ability to reject disturbances is considered.

Question: Why is the disturbance at the plant output not considered?

Explanation: Because the transfer function from the reference $r(s)$ to the output $y(s)$ is $T(s)$, the transfer function from the output disturbance $d(s)$ to the output $y(s)$ is $S(s)$, and that

$$S(s) + T(s) = 1$$

In other words, the closed-loop response and the response of the output disturbance are complementary (Figure)

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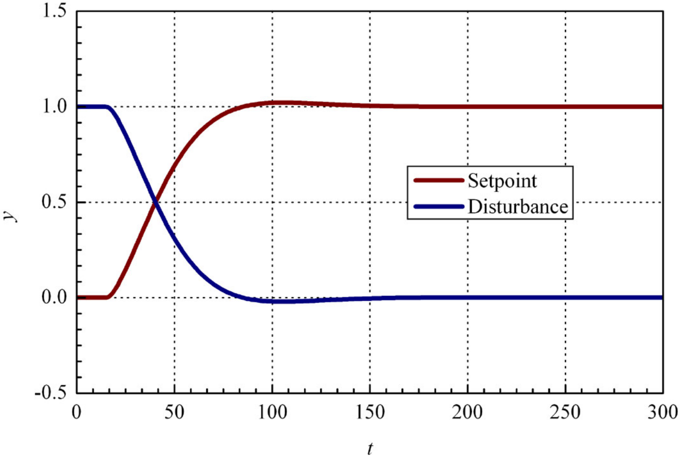

Figure: Closed-loop response and the output disturbance response

5.4 Control of Inverse Response Processes

Goal of this section: Employ a simple NMP plant to compare the features of the H_∞ controller and the H_2 controller

Feature of NMP plants: For stable plants, with the same magnitude there exist plants exhibiting less phase than NMP plants

One example:

$$G(s) = \frac{1-s}{1+s}$$

Its magnitude is $|G(j\omega)| = 1$ and its phase is $\angle G(j\omega) = \arctan 2\omega/(\omega^2-1)$. Obviously, there exist other plants with the same magnitude and less phase. For example, the magnitude of $G(s) = 1$ is 1 and the phase is 0

NMP plants are difficult to control

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Inverse Response Process

The inverse response process is a special NMP plant

An example: An inverse response may occur in a distillation column. When the steam pressure to the reboiler is suddenly increased. The initial effect is usually to increase the amount of frothing on the trays above the reboiler, causing a rapid spillover of liquid from these trays into the reboiler. This effect results in an initial increase in the rebolier liquid level. However, the increase in steam pressure ultimately will decrease the reboiler liquid level by boiling off more liquid.

Feature of the inverse response process: Its transfer function has one zero or an odd number of zeros in the open RHP

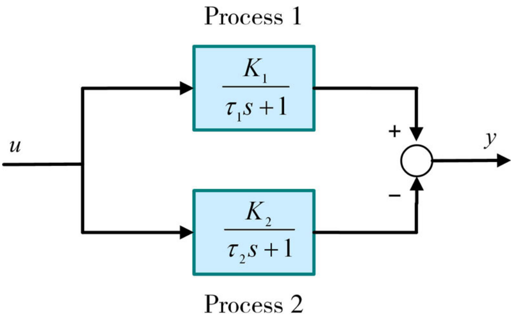

Figure: Two opposing first-order processes

The simplest inverse response process consists of two first-order plants with opposing effects. The transfer function of the whole plant is

$$G(s) = \frac{K_1}{\tau_1 s + 1} - \frac{K_2}{\tau_2 s + 1}$$

or

$$G(s) = \frac{(K_1\tau_2 - K_2\tau_1)s + (K_1 - K_2)}{(\tau_1 s + 1)(\tau_2 s + 1)}$$

Condition for the occurrence of inverse response:

$$\tau_1/\tau_2 > K_1/K_2 > 1$$

System response: The process 2 initially reacts faster than the process 1, but the process 1 ultimately reaches a higher steady state value than the process 2 (Figure). The transfer function of the plant has a zero in the open RHP:

$$z_r = \frac{K_2 - K_1}{K_1\tau_2 - K_2\tau_1} > 0$$

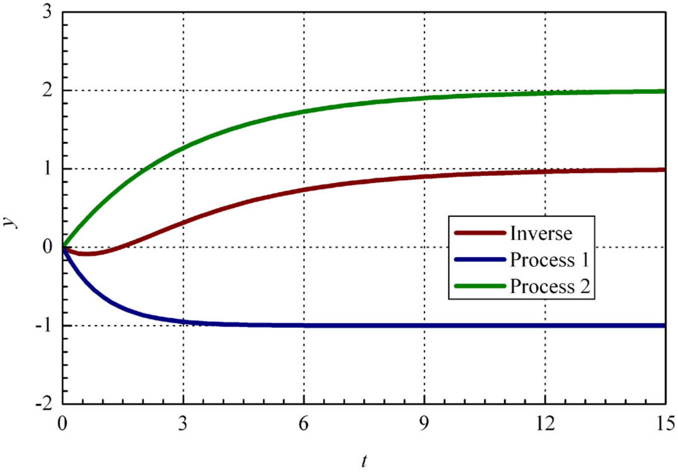

Figure: Overall response for $\tau_1/\tau_2 > K_1/K_2 > 1$

Optimal Control

H_∞ control: The worst ISE resulted from a set of energy-bounded inputs is minimized:

$$\min_{r(t)} \sup \|e(t)\|_2$$

or equivalently, the ∞-norm of the weighted sensitivity function is minimized:

$$\min \|W(s)S(s)\|_\infty$$

By Maximum Modulus Theorem, we have

$$\|W(s)S(s)\|_\infty \geq 1/z_r$$

It can be varified that

$$C(s) = \frac{1}{(K_1 - K_2)} \frac{(\tau_1 s + 1)(\tau_2 s + 1)}{\lambda^2 s^2 + (2\lambda + K_2\tau_1 - K_1\tau_2)s}$$

Thus, the H_∞ optimal solution can be reached by only a PID controller. The closed-loop transfer function is

$$T(s) = \frac{-z_r^{-1}s + 1}{(\lambda s + 1)^2}$$

Notice that no poles of the plant appears in $T(s)$. All of them are canceled by the H_∞ controller.

H_2 control: Minimizes the ISE resulted from the impulse input:

$$\min \|e(t)\|_2$$

or equivalently, the 2-norm of the weighted sensitivity function is minimized:

$$\min \|W(s)S(s)\|_2$$

Thus, the H_∞ optimal solution can be reached by only a PID controller. The closed-loop transfer function is

$$T(s) = \frac{-z_r^{-1}s + 1}{(\lambda s + 1)^2}$$

Notice that no poles of the plant appears in $T(s)$. All of them are canceled by the H_∞ controller.

H_2 control: Minimizes the ISE resulted from the impulse input:

$$\min \|e(t)\|_2$$

or equivalently, the 2-norm of the weighted sensitivity function is minimized:

$$\min \|W(s)S(s)\|_2$$

It is easy to obtain the following H_2 controller:

$$C(s) = \frac{1}{(K_1 - K_2)} \frac{(\tau_1 s + 1)(\tau_2 s + 1)}{\lambda z_r^{-1} s^2 + (2z_r^{-1} + \lambda)s}$$

which is also a PID controller. The closed-loop transfer function is

$$T(s) = \frac{-z_r^{-1} s + 1}{(z_r^{-1} s + 1)(\lambda s + 1)}$$

Factorize the plant into the MP part and the all-pass part:

$$G(s) = (K_1 - K_2) \frac{(z_r^{-1} s + 1)}{(\tau_1 s + 1)(\tau_2 s + 1)} \frac{-z_r^{-1} s + 1}{z_r^{-1} s + 1}$$

It is seen that the H_2 controller only cancels the poles in the MP part of the plant, while those poles in the all-pass part are retained

Filter

The filter that makes the controller proper is not unique

The constraint imposed on the filter: It should be a low-pass transfer function satisfying the requirement on the asymptotic tracking

If the following filter is chosen for the H_2 controller:

$$J(s) = \frac{z_r^{-1}s + 1}{(\lambda s + 1)^2}$$

then the H_2 controller will be identical to the H_∞ controller. Certainly, the H_2 controller can also be equivalent to some other controllers by selecting an appropriate filter

Side-effect: Such a filter is seldom used, since it introduces additional dynamics

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Time Domain Responses

Since there exists no time delay in the plant, the response of the closed-loop system can be easily computed. For example, the response of the H_∞ controller is

$$h(t) = 1 - \left(1 + \frac{t}{\lambda} + \frac{tz_r^{-1}}{\lambda^2}\right) e^{-t/\lambda}$$

The response does not have an overshoot. Let $dh(t)/dt = 0$. One can get the time that the peak of the inverse response happens:

$$t = \frac{\lambda}{1 + \lambda z_r}$$

Substituting this into $h(t)$ gives the peak of the inverse response:

$$1 - \frac{1 + \lambda z_r}{\lambda z_r} e^{-1/(1+\lambda z_r)}$$

Let $h(t) = 0.9$. One can get the rise time

Application Scope

On the surface:

H_2 control$-$ $>$A known specific input$-$ $>$ Limited scope

H_∞ control$-$ $>$All energy-bounded inputs$-$ $>$Much wider scope

However, **this is not the case**

Goal of introducing weights: Express the design procedure in a unified form for inputs that differ from the impulse

H_∞ optimal control: The choice of $W(s)$ corresponds with the input. Once the weighting function is determined, the most frequent inputs are assumed

Conclusion: There is no evident difference in the applying scope of the two controllers

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Similar insights comes also from the system gain. Recall the discussion in Chapter 3:

$$\|e(t)\|_{\infty} \leq \|W(s)S(s)\|_2 \, \|r(t)\|_2$$

Thus, another objective of the H_2 optimal control is to minimize the maximum magnitude of the error for energy-bounded inputs

Example

Magnetic levitation (maglev) trains may replace airplanes on routers of several hundred kilometers (Figure). In a maglev system, vehicles are suspended on a guideway above the highway and guided by magnetic forces instead of relying on wheels or aerodynamic forces. Maglev travel would be fast, operating at 500 km/hour. Ideally maglev trains can offer the environmental and safety advantages of a train and the speed of an airplane

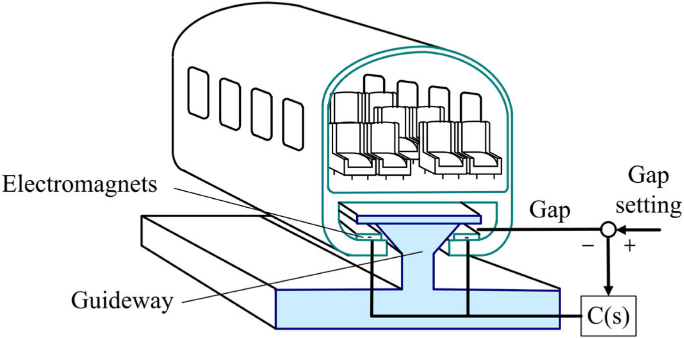

Figure: Diagram of the maglev train

Example (ctd.1)

There is an electronic *pas de deux* between the vehicle's weight and the repelling force of the electromagnets. The gap between each arm and the guideway is measured 100 000 times per second. This distance is fed to a control system, in which the current in the support magnets is continually adjusted so as to reach an equilibrium point at which the weight of the vehicle is supported by the magnet repellence. The result: the vehicle hovers and the gap between each arm and the underside of the guideway is kept between 10 ± 2mm. The dynamic model of the gap is

$$\boxed{G(s) = \frac{s-4}{(s+2)^2}.}$$

Example (ctd.2)

The H_∞ controller is

$$\boxed{C(s) = -\frac{(s+2)^2}{4\lambda^2 s^2 + (8\lambda+1)s},}$$

with the performance degree $\lambda = 1$. The H_2 controller is

$$\boxed{C(s) = -\frac{(s+2)^2}{\lambda s^2 + (4\lambda+2)s}.}$$

Its performance degree is tuned so that the two controllers have the same overshoot: $\lambda = 1.6$. A unit step reference is added at $t = 0$ and a unit step load is added at $t = 30$. The responses of the closed-loop system are shown in Figure. Although thoroughly different norms are used, the obtained responses are similar

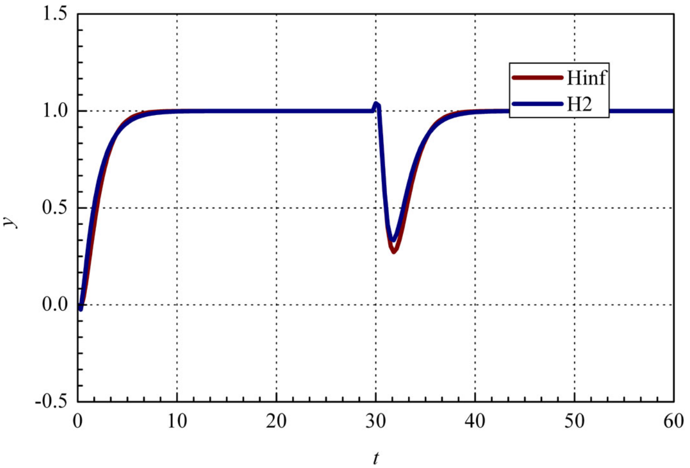

Figure: Responses of the gap control system

5.5 PID Controllers Based on the Maclaurin Series Expansion

Preceding designs: Employ the rational approximation to expand the time delay, and then the controller was designed for the approximate plant

This section: The desired controller is first designed for the original plants with time delay, and then a PID controller is derived by the rational approximation of the obtained overall controller

Two important features:

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2. It can be directly used for high-order plants

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Two important features:

1. It can provide better performance
2. It can be directly used for high-order plants

Consider a stable MP plant described by

$$G(s) = \frac{KN_-(s)}{M_-(s)} e^{-\theta s}$$

where $N_-(s)$ and $M_-(s)$ are the polynomials with roots in the LHP, $N_-(0) = M_-(0) = 1$, and $\deg\{N_-(s)\} \leq \deg\{M_-(s)\}$. The desired closed-loop transfer function is chosen as follows:

$$T(s) = \frac{e^{-\theta s}}{(\lambda s + 1)^{n_j}}$$

where $n_j = \deg\{M_-(s)\} - \deg\{N_-(s)\}$ for strictly proper plants and $n_j = 1$ for bi-proper plants. In the next chapter, it will be proved that the desired closed-loop transfer function is suboptimal. Since $Q(s) = T(s)/G(s)$ is stable, the closed-loop system is internally stable. Then the controller is

$$C(s) = \frac{1}{G(s)} \frac{T(s)}{1 - T(s)}$$

Here, only **stable MP** plants are considered. The design procedure for NMP plants is similar

The problem now reduces to that find a PID controller that approximates the desired controller

A method: Use the Maclaurin series expansion

Since

$$\lim_{s \to 0} \left[(\lambda s + 1)^{n_j} - e^{-\theta s} \right] = 0$$

$C(s)$ has a pole at the origin. Expanding $C(s)$ in a Maclaurin series gives

$$C(s) = \frac{f(s)}{s}$$

with

$$f(s) = f(0) + f'(0)s + \frac{f''(0)}{2!}s^2 + \dots$$

The first three terms are taken to approximate the desired controller. The three terms form a PID controller:

$$C(s) = K_C\left(1 + \frac{1}{T_I s} + T_D s\right)$$

where

$$K_C = f'(0), \quad T_I = f'(0)/f(0), \quad T_D = \frac{f''(0)}{2f'(0)}$$

Certainly, one can also take the first two terms to form a PI controller.

Now let us see how to compute the controller parameter with a given plant. For convenience of presentation, let

$$N(s) = \frac{M_-(s)}{KN_-(s)}$$

$$M(s) = \frac{(\lambda s + 1)^{n_j} - e^{-\theta s}}{s}$$

The values of $f(s)$ and its first-order and second-order derivatives at the origin are

$$f(0) = \frac{N(0)}{M(0)}$$

$$f'(0) = \frac{N'(0)M(0) - M'(0)N(0)}{M(0)^2}$$

$$f''(0) = \frac{\begin{array}{c} N''(0)M(0)^2 - M''(0)N(0)M(0) - \\ 2M'(0)N'(0)M(0) + 2M'(0)^2N(0) \end{array}}{M(0)^3}$$

If the plant is of first-order; that is,

$$G(s) = \frac{Ke^{-\theta s}}{\tau s + 1}$$

we have

$$N(0) = \frac{1}{K},\ N'(0) = \frac{\tau}{K},\ N''(0) = 0$$
$$M(0) = \lambda + \theta,\ M'(0) = -\frac{\theta^2}{2},\ M''(0) = \frac{\theta^3}{3}$$

The function $f(s)$ and its first and second derivatives at the origin are given by

$$f(0) = \frac{1}{K(\lambda + \theta)}$$
$$f'(0) = \frac{\theta^2 + 2\lambda\tau + 2\theta\tau}{2K(\lambda + \theta)^2}$$
$$f''(0) = \frac{\theta^2(-2\lambda\theta + \theta^2 + 6\lambda\tau + 6\theta\tau)}{6K(\lambda + \theta)^3}$$

Consequently, the PID controller parameters are

$$T_I = \tau + \frac{\theta^2}{2(\lambda+\theta)}$$
$$K_C = \frac{T_I}{K(\lambda+\theta)}$$
$$T_D = \frac{\theta^2(3T_I-\theta)}{6T_I(\lambda+\theta)}$$

When a second-order model is used:

$$G(s) = \frac{Ke^{-\theta s}}{(\tau_1 s+1)(\tau_2 s+1)}$$

Utilizing the Maclaurin series expansion, we have

$$N(0) = \frac{1}{K},\ N'(0) = \frac{\tau_1+\tau_2}{K},\ N''(0) = \frac{2\tau_1\tau_2}{K}$$
$$M(0) = 2\lambda+\theta,\ M'(0) = \lambda^2 - \frac{\theta^2}{2},\ M''(0) = \frac{\theta^3}{3}$$

The function $f(s)$ and its first and second derivatives are given by

$$f(0) = \frac{1}{K(2\lambda + \theta)}$$

$$f'(0) = \frac{-2\lambda^2 + \theta^2 + 2(2\lambda + \theta)(\tau_1 + \tau_2)}{2K(2\lambda + \theta)^2}$$

$$f''(0) = \frac{\begin{array}{c} 2\tau_1\tau_2(2\lambda + \theta)^2 - \theta^3(2\lambda + \theta)/3 - \\ (\tau_1 + \tau_2)(2\lambda^2 - \theta^2)(2\lambda + \theta) + 2(\lambda^2 - \theta^2/2)^2 \end{array}}{K(2\lambda + \theta)^3}$$

The PID controller parameters are

$$T_I = \tau_1 + \tau_2 - \frac{2\lambda^2 - \theta^2}{2(2\lambda + \theta)}, K_C = \frac{T_I}{K(2\lambda + \theta)}$$

$$T_D = T_I - \tau_1 - \tau_2 + \frac{12\tau_1\tau_2\lambda + 6\tau_1\tau_2\theta - \theta^3}{T_I(12\lambda + 6\theta)}$$

Problems: The integral and derivative constants computed based on the above formulas might be negative for some plants

Two solving methods:

- The designer can take only the first two terms to form a PI controller
- Take the first four terms to form a controller

Tuning: The effect of the performance degree on the closed-loop response is similar to that in the H_∞ PID controller and the H_2 PID controller

Performance: Since more complicated formulas are used to compute PID controller parameters, it is not surprised that the Maclaurin PID controller can provide better performance than the H_∞ PID controller and the H_2 PID controller.

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5.6 PID Controllers with the Best Achievable Performance

An important problem: Explore the performance limit of a controller

This section discusses the problem for PID controllers. More precisely, the following problems are studied:

1. What a performance limit does the PID controller have?
2. Is it possible to analytically design the PID controller with best achievable performance?
3. How can the PID controller be tuned for quantitative performance and robustness?

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Two different design procedures have been developed in the foregoing sections:

1. The PID controller is designed by reducing the order of the plant
2. The PID controller is designed by employing the Maclaurin series expansion to reduce the order of the desired controller

An improvement on the performance is possible

Since the Pade approximation can provide higher precision than the Maclaurin series expansion, the PID controller will be designed in this section by employing the Pade approximation for controller reduction. The property of the Pade approximation guarantees that the resulting controller provides the **best** performance that a PID controller can achieve among current **analytical** design methods

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The solving of the problem follows the design procedure of the Maclaurin PID controller. First, expand the desired controller $C(s)$:

$$C(s) = \frac{f(s)}{s} = \frac{1}{s}\left[f(0) + f'(0)s + \frac{f''(0)}{2!}s^2 + \frac{f^{(3)}(0)}{3!}s^3 + ...\right]$$

As we know, the ideal PID controller has a pure derivative term in it and thus is not physically realizable. Realizable PID controllers are usually in three forms, which have been listed in Section **??**. All of the three can be expressed in a unified form:

$$C(s) = \frac{a_2 s^2 + a_1 s + a_0}{s(b_1 s + 1)}$$

where a_0, a_1, a_2 and b_1 are positive real numbers. Let the Pade approximation of $f(s)$ be

$$f(s) = \frac{a_2 s^2 + a_1 s + a_0}{b_1 s + 1}$$

Then, we have

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} f(0) & 0 \\ f'(0) & f(0) \\ f''(0)/2! & f'(0) \end{bmatrix} \begin{bmatrix} 1 \\ b_1 \end{bmatrix}$$

$$b_1 f''(0)/2! = -f^{(3)}(0)/3!$$

It follows that

$$\begin{aligned} a_0 &= f(0) \\ a_1 &= b_1 f(0) + f'(0) \\ a_2 &= b_1 f'(0) + f''(0)/2! \\ b_1 &= -\frac{f^{(3)}(0)}{3f''(0)} \end{aligned}$$

Assume that the PID controller is in the form of

$$C = K_C\left(1 + \frac{1}{T_I s} + T_D s\right)\frac{1}{T_F s + 1}$$

The controller parameters are

$$K_C = a_1, \quad T_I = \frac{a_1}{a_0}, \quad T_D = \frac{a_2}{a_1}, \quad T_F = b_1$$

Disadvantage of the Pade approximation: It may be unstable even if the original transfer function is stable

Solving methods:

1. If the controller is required to be a PID controller whose order is less than three. One could choose the first two or three terms of the Maclaurin series expansion as a PID controller
2. If the designer does not have a strict requirement on the controller form, one could choose a PID controller with a second-order lag or a higher order controller

Consider the following first-order plant with time delay:

$$G(s) = \frac{Ke^{-\theta s}}{\tau s + 1}$$

The function $f(s)$ and its derivatives are given by

$$f(0) = \frac{1}{K(\lambda + \theta)}$$

$$f'(0) = \frac{\theta^2 + 2\lambda\tau + 2\theta\tau}{2K(\lambda + \theta)^2}$$

$$f''(0) = \frac{\theta^2(-2\lambda\theta + \theta^2 + 6\lambda\tau + 6\theta\tau)}{6K(\lambda + \theta)^3}$$

$$f^{(3)}(0) = \frac{\theta^3(-2\tau\lambda\theta + 2\tau\theta^2 - 4\lambda^2\tau - 2\theta^2\lambda + \theta\lambda^2)}{4K(\lambda + \theta)^4}$$

Parameters of the PID controller are

$$a_0 = \frac{1}{K(\lambda+\theta)}$$

$$a_1 = \frac{\theta^3 + 6\tau\theta^2 - \theta^2\lambda + 12\tau^2\theta + 12\tau^2\lambda}{2K(-2\lambda\theta + \theta^2 + 6\tau\lambda + 6\tau\theta)(\lambda+\theta)}$$

$$a_2 = \frac{\theta(\theta^3 + 6\tau\theta^2 + 24\tau^2\theta - 6\tau\theta\lambda + 24\tau^2\lambda)}{12K(-2\lambda\theta + \theta^2 + 6\tau\lambda + 6\tau\theta)(\lambda+\theta)}$$

$$b_1 = -\frac{\theta(-2\tau\lambda\theta + 2\tau\theta^2 - 4\lambda^2\tau - 2\theta^2\lambda + \theta\lambda^2)}{2(-2\lambda\theta + \theta^2 + 6\tau\lambda + 6\tau\theta)(\lambda+\theta)}$$

While the above PID controller provides the best achievable performance as compared with the H_∞ controller and the H_2 controller, the corresponding formulas are in the most complicated form

5.7 Choice of the Filter

Function of the Filter

1. The optimal controller $Q_{opt}(s)$ is usually improper. One main function of the filter is to make $Q_{opt}(s)$ proper. Certainly, the controller becomes suboptimal after the filter is introduced.
2. Since $S(s) = 1 - G(s)Q(s)$ and $T(s) = G(s)Q(s)$, the filter parameter can be utilized to tune the nominal performance and the robustness, and to quantitatively trade off between the two objectives.
3. There is a direct relationship between the filter parameter and the control variable, since $u(s) = Q(s)r(s)$. If the control structure cannot be modified, one can confine the magnitude of the control variable by adjusting the filter parameter.

How to Choose the Filter

The filter should at least satisfy the following requirements:

1. The closed-loop system is internally stable.
2. The controller $Q(s) = Q_{opt}(s)J(s)$ is proper.
3. Asymptotic tracking is achieved.

The first condition is easy to satisfy. If the plant is stable, the closed-loop system is internally stable as long as $Q(s)$ is stable.

The second condition can also be easily achieved. As we know, an improper transfer function implies that the degree of its numerator is greater than that of its denominator. To make it proper, one can simply introduce a filter with its numerator degree less than its denominator degree. As the filter is stable, it is of low-pass.

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Now consider the third condition. As a basic requirement on the closed-loop performance of control systems, the tracking error should vanish asymptotically. Recall that for asymptotic tracking a Type m system should satisfy

$$\lim_{s\to 0} \frac{1 - G(s)Q_{opt}(s)J(s)}{s^k} = 0, \quad k = 0, 1, ..., m-1$$

or

$$\lim_{s\to 0} \frac{d^k}{ds^k}[1 - G(s)Q_{opt}(s)J(s)] = 0, \quad k = 0, 1, ..., m-1$$

However, these conditions are still not enough for determining the structure and the parameter of a filter. For example, one can choose a filter with either single parameter or with multiple parameters and multiple zeros

Function of zeros: Can be utilized to satisfy some special design objectives, such as tracking a complex input

Side effect of zeros: Zeros are seldom introduced to the filter unless it is necessary, since this will change performance in a way difficult to grasp and may limit the performance as well

The usual structure of a filter consists of one or more first-order lags in series. To simplify the design task, usually there is only one parameter in the filter:

$$J(s) = \frac{\beta_{m-1}s^{m-1} + \ldots + \beta_1 s + \beta_0}{(\lambda s + 1)^{n_j}}$$

where λ is the performance degree, n_j should be chosen large enough to make $Q(s) = Q_{opt}(s)J(s)$ proper, for a stable plant m equals the number of poles that the input has at the origin, and β_i $(i = 0, 1, \ldots, m-1)$ are chosen to satisfy the requirement for asymptotic tracking

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If the plant is stable, $m = 1$ for a step input. Since

$$\lim_{s \to 0} [1 - G(s)Q_{opt}(s)J(s)] = 0$$

$\beta_0 = 1$

In the H_2 optimal control of a stable rational MP plant, $G(s)Q_{opt}(s) = 1$. Hence, the Type 1 filter is as follows:

$$J(s) = \frac{1}{(\lambda s + 1)^{n_j}}$$

Such a system can track inputs of step type without offset. If the input is a ramp, then a Type 2 filter must be used:

$$J(s) = \frac{n_j \lambda s + 1}{(\lambda s + 1)^{n_j}}$$

Typical responses of filters are shown in Figure

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Typical responses of filters are shown in Figure

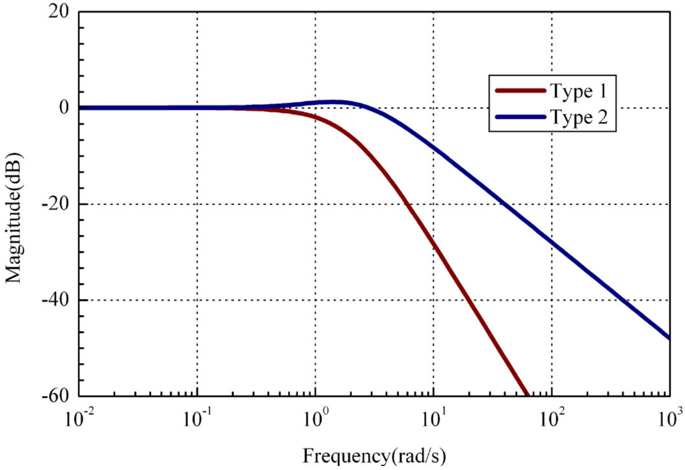

Figure: Typical response of the filter

Order of the filter: However, the higher the order, the more complicated the controller. Due to this reason, the order of the filter should be chosen such that $Q(s)$ **is bi-proper** for a strictly proper plant, or the degree of its denominator is higher by one than that of its numerator for a bi-proper plant

Performance degree: For an improved model, a better performance can be obtained by decreasing the performance degree. When the uncertainty increases, one has to increase the performance degree to obtain better robustness. In this way, a reasonable tradeoff between the two competing objectives can easily be achieved

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Constraint on the Control Variable

Objective of H_2 control:

$$\min \int_0^\infty e^2(t)dt$$

Two internally related problems:

- The controller is improper, which is not physically realizable
- The magnitude peak of the control variable is usually large

Solving method: Use a suboptimal controller.

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For example, one can use the performance index

$$\min \int_0^\infty [Qe^2(t) + Ru^2(t)]dt$$

where Q and R are the constant weights. The optimal controller designed by this index is **suboptimal** for the original ISE index

Shortcoming of the design: It is not known how to determine the weights; moreover, the computation complexity is relatively high

In some literature, Q and R are simply taken as 1:

$$\min \int_0^\infty [e^2(t) + u^2(t)]dt$$

Evidently, this is not a good choice, since the system performance depends on the weight

Solution of this book: Choose an appropriate filter $J(s)$. This works because $u(s) = Q_{opt}(s)J(s)r(s)$

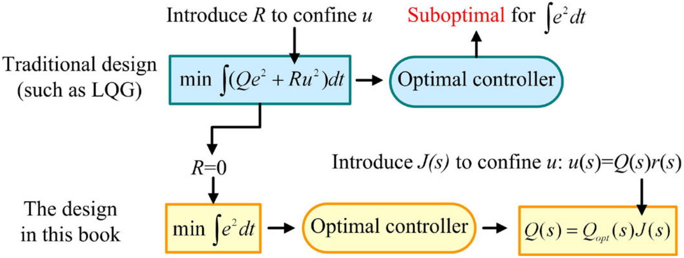

Figure: Two different optimizing procedures

End of Chapter 5