Chapter 3 Essentials of Robust Control

3.1 Norms and System Gains

Why Do We Need Norms

The performance of a control system is usually specified in terms of the "size" of certain signals.

Solution: The "size" of a signal can be defined by introducing norms. "The signal is small" means its norm is small

Consider a signal $r(t)$. A norm is a nonnagative real number, denoted by $\|r(t)\|$, that satisfying the following properties:

1. $\|r(t)\| = 0$ if and only if $r(t) = 0$, $\forall t$.
2. $\|\alpha r(t)\| = |\alpha| \|r(t)\|$, α is any real number.
3. $\|r_1(t) + r_2(t)\| \leq \|r_1(t)\| + \|r_2(t)\|$.

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Frequently Used Signal Norms

1-norm. The 1-norm of $r(t)$ is the integral of its absolute value:

$$\|r(t)\|_1 := \int_{-\infty}^{\infty} |r(t)|\, dt$$

2-norm. The 2-norm of $r(t)$ is

$$\|r(t)\|_2 := \left[\int_{-\infty}^{\infty} r^2(t)\, dt\right]^{1/2}$$

Suppose that $r(t)$ is the current through a 1Ω resistor. The instantaneous power equals $r(t)^2$ and the energy equals $\|r(t)\|_2^2$

∞-norm. The ∞-norm of $r(t)$ is the least upper bound of its absolute value:

$$\|r(t)\|_\infty := \sup_t |r(t)|$$

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Frequently Used System Norms

Consider a linear time-invariant and causal system $T(t)$, of which the input is $r(t)$ and the output is $y(t)$:

$$\begin{aligned} y(t) &= T(t) * r(t) \\ &= \int_{-\infty}^{\infty} T(t-\tau) r(\tau)\, d\tau \end{aligned}$$

Let $T(s)$ denote the transfer function of $T(t)$. Norms can also be defined for the system $T(s)$:

2-norm.

$$\|T(s)\|_2 := \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} |T(j\omega)|^2\, d\omega \right]^{1/2}$$

∞-norm.

$$\|T(s)\|_\infty := \sup_\omega |T(j\omega)|$$

Theorem

The 2-norm of $T(s)$ is finite if and only if $T(s)$ is strictly proper and has no poles on the imaginary axis. The ∞-norm of $T(s)$ is finite if and only if $T(s)$ is proper and has no poles on the imaginary axis.

Proof.

Assume that $T(s)$ is strictly proper and has no poles on the imaginary axis. Then the Bode magnitude plot rolls off at high frequencies. It is not hard to see that the plot of $c/(\tau s+1)$ is higher than that of $T(s)$ for sufficiently large positive c and sufficiently small positive τ, but the 2-norm of $c/(\tau s+1)$ equals $c/\sqrt{2\tau}$. Hence $T(s)$ has finite 2-norm.

The rest of the proof follows similar lines. $\square$

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Computation of System Norms

Suppose that $T(s)$ is strictly proper and has no poles on the imaginary axis. We have

$$
\begin{aligned}
\|T(s)\|_2^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty}|T(j\omega)|^2\,d\omega \\
&= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}T(-s)T(s)\,ds \\
&= \frac{1}{2\pi j}\oint T(-s)T(s)\,ds
\end{aligned}
$$

By residue theorem, $\|T(s)\|_2^2$ equals the sum of the residues of $T(-s)T(s)$ at its poles in the LHP.

Example

Take $T(s) = 1/(s+1)$. The LHP pole of $T(-s)T(s)$ is at $s = -1$. The residue at this pole equals

$$\lim_{s \to -1} (s+1) \frac{1}{-s+1} \frac{1}{s+1} = \frac{1}{2}$$

Hence $\|T(s)\|_2 = 1/\sqrt{2}$

The ∞-norm can be computed by search. Choose a series of frequency points for $T(s)$:

$$\{\omega_1, \ldots, \omega_n\}$$

The estimate for $\|T(s)\|_\infty$ is

$$\max_{1 \le k \le n} |T(j\omega_k)|$$

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Properties of System Norms

Property 1 of 2-norm:

If $T(s)$ is stable, by Parseval's theorem:

$$\|T(s)\|_2 = \|T(t)\|_2$$

Property 2 of 2-norm:

Assume that the complex conjugate of $c = a + bi$ is $\bar{c} = a - bi$

Theorem

If $T_1(s)$ does not have poles in Re $s > 0$ while $T_2(s)$ does not have poles in Re $s < 0$, then

$$\|T_1(s) + T_2(s)\|_2^2 = \|T_1(s)\|_2^2 + \|T_2(s)\|_2^2$$

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Proof.

$$
\begin{aligned}
& \| T_1(s) + T_2(s) \|_2^2 \\
= \; & \frac{1}{2\pi} \int |T_1(j\omega) + T_2(j\omega)|^2 \, d\omega \\
= \; & \| T_1(s) \|_2^2 + \| T_2(s) \|_2^2 + 2\mathrm{Re}\left[\frac{1}{2\pi} \int \overline{T_1(j\omega)} T_2(j\omega) d\omega \right].
\end{aligned}
$$

Now, it suffices to show the last integral equals zero. Convert it into a contour integral by closing the imaginary axis with an infinite radius semicircle in the LHP:

$$
\frac{1}{2\pi} \int \overline{T_1(j\omega)} T_2(j\omega) d\omega = \frac{1}{2\pi j} \oint T_1(-s) T_2(s) ds.
$$

Proof (ctd.1).

Recall Cauchy's theorem, which concludes that if a function does not have poles in a simply connected region, then its integral on any closed contour contained in the region equals zero. Therefore, the right-hand side of the above equation equals zero. $\square$

The ∞-norm of $T(s)$ equals the distance from the origin to the farthest point on the Nyquist plot of $T(s)$. It is also appears as the peak on the Bode magnitude plot of $T(s)$

Property of ∞-norm:

It is sub-multiplicative:

$$\|T_1(s)T_2(s)\|_\infty \leq \|T_1(s)\|_\infty \|T_2(s)\|_\infty$$

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System Gains

An important question in design: If it is known how large the input is, how large is the output going to be? For example, the input is a signal with its 2-norm less than or equal to 1. What is the least upper bound on the 2-norm of the output?
Answer: The answer to this question correlates with an important concept called **system gain**

Table: System gains for SISO systems

	$r(t)=\delta(t)$	$\|r(t)\|_2$
$\|y(t)\|_2$	$\|T(s)\|_2$	$\|T(s)\|_\infty$
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Entry (1,1)

The entry shows that the energy of the output is the square of the 2-norm of the system transfer function when the input is an impulse.

Proof.

If $r(t) = \delta(t)$, then

$$y(t) = \int_{-\infty}^{\infty} T(t-\tau)\delta(\tau)\, d\tau = T(t)$$

Therefore, $\|y(t)\|_2 = \|T(t)\|_2$. As we know, $\|T(t)\|_2 = \|T(s)\|_2$ □

Entry (2,1)

Proof.

Again, since $y(t) = T(t)$. □

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Proof.

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Entry (1,2)

The entry shows that the maximum energy of the output is the square of the ∞-norm of the system transfer function when the input is a signal of which the energy is bounded by unity

Proof.

First, $\|T(s)\|_\infty$ is an upper bound on the system gain:

$$\begin{aligned}
\|y(t)\|_2^2 &= \|y(s)\|_2^2 \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} |T(j\omega)|^2 |r(j\omega)|^2 \, d\omega \\
&\leq \|T(s)\|_\infty^2 \frac{1}{2\pi}\int_{-\infty}^{\infty} |r(j\omega)|^2 \, d\omega \\
&= \|T(s)\|_\infty^2 \|r(s)\|_2^2 \\
&= \|T(s)\|_\infty^2 \|r(t)\|_2^2
\end{aligned}$$

Proof (ctd.1).

To show that $\|T(s)\|_\infty$ is the least upper bound, it is enough to prove that the identity holds for at least one input. Choose a frequency ω_o where

$$|T(j\omega_o)| = \|T(s)\|_\infty$$

Similar to the time domain impulse, construct a frequency domain impulse signal $\delta_f(j\omega)$:

$$|\delta_f(j\omega)| = \begin{cases} \infty & \omega = \omega_o \\ 0 & \omega \neq \omega_o \end{cases}$$

with

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |\delta_f(j\omega)|^2 \, d\omega = 1$$

Proof (ctd.2).

For this specific input we have

$$\begin{aligned}
\|y(t)\|_2^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |T(j\omega)|^2 |\delta_f(j\omega)|^2 \, d\omega \\
&= |T(j\omega_o)|^2 \\
&= \|T(s)\|_\infty^2
\end{aligned}$$

□

There exists no ideal frequency domain impulse in a real system. However, the impulse can be approximated by

$$|\delta_f(j\omega)| = \begin{cases} \sqrt{\pi/2\epsilon} & |\omega - \omega_o| < \epsilon \text{ or } |\omega + \omega_o| < \epsilon \\ 0 & \text{else} \end{cases}$$

where ϵ is a small positive number

Proof (ctd.2).

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Entry (2,2)

The entry shows that the maximum amplitude of the output is the 2-norm of the system transfer function when the input is a signal of which the energy is bounded by unity

Proof.

With the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
|y(t)| &= \left| \int_{-\infty}^{\infty} T(t-\tau) r(\tau) d\tau \right| \\
&\leq \left(\int_{-\infty}^{\infty} T^2(t-\tau)\, d\tau \right)^{1/2} \left(\int_{-\infty}^{\infty} r^2(\tau)\, d\tau \right)^{1/2} \\
&= \|T(t)\|_2 \|r(t)\|_2 \\
&= \|T(s)\|_2 \|r(t)\|_2
\end{aligned}
$$

Proof (ctd. 1).

Hence

$$\|y(t)\|_\infty \leq \|T(s)\|_2.$$

To show that $\|T(s)\|_2$ is the least upper bound, apply the input

$$r(t) = T(-t)/\|T(t)\|_2$$

Then $\|r(t)\|_2 = 1$ and

$$\begin{aligned} |y(0)| &= \left|\int_{-\infty}^{\infty} T(-\tau)\frac{T(-\tau)}{\|T(t)\|_2}d\tau\right| \\ &= \frac{1}{\|T(t)\|_2}\left|\int_{-\infty}^{\infty} T^2(-\tau)d\tau\right| \\ &= \|T(t)\|_2 \end{aligned}$$

□

3.2 Internal Stability and Performance

Internal Stability

In a control system, it cannot be tolerated that a small disturbance at one location leads to unbounded signals at some other locations. It is not enough to look only at the closed-loop transfer function. To guarantee bounded internal signals, the closed-loop system must be internally stable.

Definition

A linear time-invariant control system is internally stable if the transfer functions between any two points of the system are stable.

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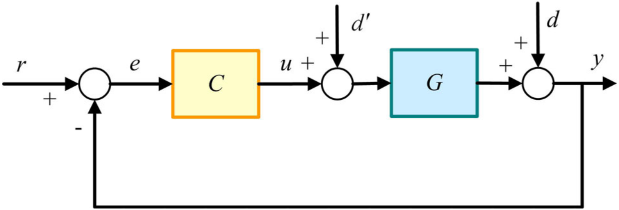
Test for Internal Stability



In a control system every two points can be selected for disturbance signal injection and observation, but some of the choices are equivalent for checking internal stability. In the unity feedback control system, there are only two **independent** outputs and two **independent** inputs. One can choose $r(s)$ and $d'(s)$ as inputs and $y(s)$ and $u(s)$ as outputs

The closed-loop system is internally stable if and only if all elements in $\mathbf{H}(s)$ are stable:

$$\begin{bmatrix} y(s) \\ u(s) \end{bmatrix} = \mathbf{H}(s) \begin{bmatrix} r(s) \\ d'(s) \end{bmatrix}$$

where

$$\mathbf{H}(s) = \begin{bmatrix} \dfrac{G(s)C(s)}{1+G(s)C(s)} & \dfrac{G(s)}{1+G(s)C(s)} \\ \dfrac{C(s)}{1+G(s)C(s)} & \dfrac{-G(s)C(s)}{1+G(s)C(s)} \end{bmatrix}$$

Compared with the concept of internal stability, the concept of stability is not a complete one, because the **RHP zero-pole cancellation** in the feedback loop is not considered.

Zero-pole cancellation: A zero and a pole are at the same point

- Some cancellations are removable, like $(s-1)/(s-1)=1$. It is unstable before the cancellation is removed
- Some cancellations are not removable, like $(e^{-s}-1)/s$. Normally, they appear in systems with time delays

Example

This example is used to illustrate the difference between the stability and the internal stability. Take

$$C(s)=\frac{s-1}{s+1}, G(s)=\frac{1}{(s-1)(s+1)}$$

It is easy to verify that the transfer function from $r(s)$ to $y(s)$ is stable, but the one from $d'(s)$ to $y(s)$ is not. Therefore, the system is internally unstable

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Performance

Sensitivity function:

$$S(s) := \frac{1}{1 + G(s)C(s)}$$

which expresses the effect of $d(s)$ on $y(s)$, or the effect of $r(s)$ on $e(s)$:

$$S(s) = \frac{y(s)}{d(s)} = \frac{e(s)}{r(s)}$$

The origin of the name:

$$\lim_{\Delta G(s)\to 0} \frac{\Delta T(s)/T(s)}{\Delta G(s)/G(s)} = \frac{dT(s)}{dG(s)} \frac{G(s)}{T(s)} = S(s)$$

Complementary sensitivity function (i.e.closed-loop transfer function):

$$T(s) = \frac{G(s)C(s)}{1 + G(s)C(s)}$$

Nominal plant: The exact model. One can take the "center" of all uncertain plants as the nominal plant

Desire on $S(s)$: $|S(j\omega)|$ **is as small as possible**. A smaller $|S(j\omega)|$ implies that the changes of the reference and the disturbance have less effect on the system output. Such a system has a better ability on disturbance rejection.

Perfect control: $|S(j\omega)|=0$ for all frequencies

Perfect control is **impossible**, since for strictly proper plants

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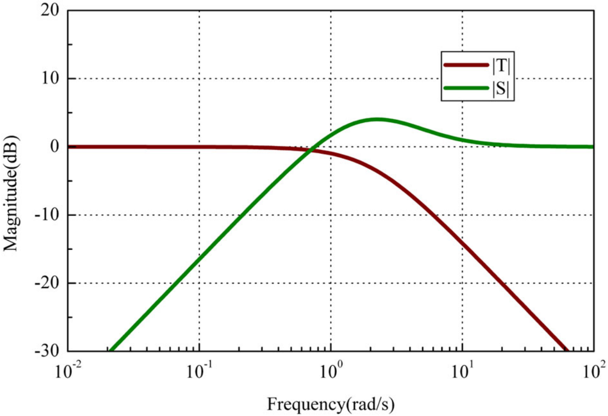
Desrie on $T(s)$: $|T(j\omega)|$ should be made as close as possible to unity. A $|T(j\omega)|$ close to unity means that the system has a large bandwidth and thus has a good tracking ability

Constraint: As $T(s) + S(s) = 1$, $|T(j\omega)|$ can be made equal to unity only within a finite frequency range



Performance

Basic objective: Keep the error between the plant output $y(t)$ and the reference $r(t)$ small when the overall system is affected by the external disturbance and the uncertainty of the plant

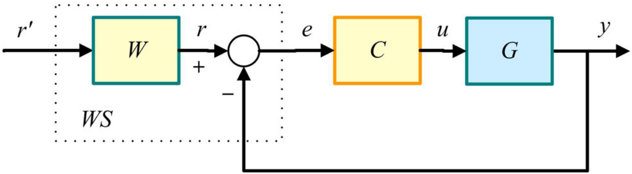
To quantify performance, an index of “smallness” for the error has to be defined

$\mathbf{H}_2$ index:

$$\min \int_0^\infty e^2(t)dt = \min \|e(t)\|_2^2$$

The **structure** and **parameters** of controller can be determined from the solution of the above optimization problem

If $r(s)$ is known, a weighting function $W(s) = r(s)$ is introduced to normalize the reference so that the system input $r'(s)$ is an impulse



Use Table 3.1.1: If the input is an impulse, the energy of the output $e(t)$ is the square of the 2-norm of the system transfer function:

$$\|e(t)\|_2 = \|W(s)S(s)\|_2, \quad \left\|\frac{r(s)}{W(s)}\right\|_2 = 1$$

Hence, the H_2 performance index in Laplace domain is

$$\min\|W(s)S(s)\|_2$$

Assume that $\min \|W(s)S(s)\|_2 \to \epsilon$. Include the effect of ϵ into $W(s)$; the index can also be expressed as

$$\|W(s)S(s)\|_2 < 1$$

H_∞ index:

$$\min_{r(t)} \sup \int_0^\infty e^2(t)dt = \min_{r(t)} \sup \|e(t)\|_2^2$$

Utilize Table 3.1.1: If the input is a signal of which the energy is bounded by unity, the maximum energy of the output is the square of the ∞-norm of the system transfer function:

$$\sup_{r(t)} \|e(t)\|_2 = \|W(s)S(s)\|_\infty, \quad \left\| \frac{r(s)}{W(s)} \right\|_2 \leq 1$$

The design index in frequency domain can be written as

$$\min \|W(s)S(s)\|_\infty \quad \text{or} \quad \|W(s)S(s)\|_\infty < 1$$

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H$_\infty$ index:

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Why weighting functions: Treat the design problem in a unitary mathematical form. Then the system gain in Table 3.1.1 can be applied

The choice of the norm is not crucial. Normally, the choice of the performance index is not crucial neither. The trade-off inherent in the control system means that although the norms or the performance indices may differ to a large extent, the obtained responses are not very different. The design requirement can be achieved by using different norms and performance indices. Comparatively, it might be more important to choose a norm or a performance index that is mathematically convenient

Equivalence between the regulator problem and the servomechanism problem: Because $S(s) = 1 - T(s)$, to minimize $|S(j\omega)|$ means to make $|T(j\omega)|$ close to 1

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The choice of the norm is not crucial. Normally, the choice of the performance index is not crucial neither. The trade-off inherent in the control system means that although the norms or the performance indices may differ to a large extent, the obtained responses are not very different. The design requirement can be achieved by using different norms and performance indices. Comparatively, it might be more important to choose a norm or a performance index that is mathematically convenient

Equivalence between the regulator problem and the servomechanism problem: Because $S(s) = 1 - T(s)$, to minimize $|S(j\omega)|$ means to make $|T(j\omega)|$ close to 1

Asymptotic Tracking Property

Theorem

Assume that the closed-loop system is internally stable and is of Type m. Then the sensitivity function satisfies

$$\lim_{s\to 0} \frac{S(s)}{s^k} = 0, \quad k = 0, 1, ..., m-1.$$

As $t \to \infty$ the closed-loop system tracks perfectly references of the form $\sum_{k=0}^{m} a_k s^{-k}$, where a_k are real constants.

Proof.

With $S(s) = 1/[1 + L(s)]$, the result follows directly from Final Value Theorem. □

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In particular, for step and ramp inputs there are the following corollaries.

Corollary

Assume that the closed-loop system is internally stable.

1. *If the reference is a step, then the tracking error tends to be zero as* $t \to \infty$ *if and only if* $S(s)$ *has at least one zero at the origin.*
2. *If the reference is a ramp, then the tracking error tends to be zero as* $t \to \infty$ *if and only if* $S(s)$ *has at least two zeros at the origin.*

3.3 Controller Parameterization

Assume that the plant $G(s)$ is **stable**. Introduce $Q(s)$, which denotes the transfer function from $r(s)$ to $u(s)$:

$$Q(s) = \frac{C(s)}{1 + G(s)C(s)}$$

$C(s)$ can be obtained through the inverse relationship:

$$C(s) = \frac{Q(s)}{1 - G(s)Q(s)}$$

$\mathbf{H}(s)$ in the last section can be rewritten as

$$\mathbf{H}(s) = \begin{bmatrix} G(s)Q(s) & [1 - G(s)Q(s)]\,G(s) \\ Q(s) & -G(s)Q(s) \end{bmatrix}$$

Then the system is internally stable if and only if $Q(s)$ is stable.

We in fact proved the following theorem.

Theorem (Youla parameterization)

Assume that $G(s)$ is stable. All stabilizing controllers can be expressed as

$$C(s) = \frac{Q(s)}{1 - G(s)Q(s)}$$

where $Q(s)$ is any stable transfer function.

A simple explanation: Let
$G(s)$—The nominal plant
$\widetilde{G}(s)$—The real plant
By **equivalently** transforming the unity feedback control loop, one can obtain the IMC structure

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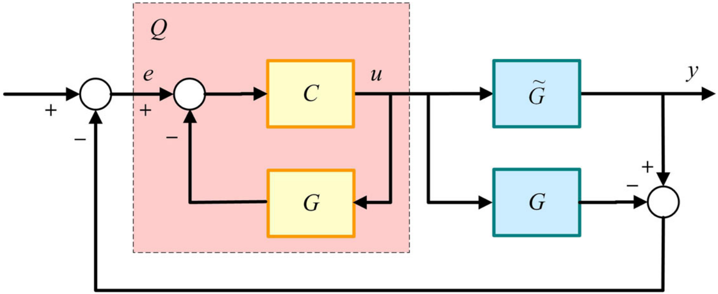

Figure: IMC structure

$Q(s)$—IMC controller
Assume that the model is exact, that is, $G(s) = \widetilde{G}(s)$. The figure shows that the stability of the closed-loop system is determined only by $Q(s)$. This is the conclusion given by the Youla parameterization

Remarks:

- If $G(s)$ is viewed as a reference model, model reference control can be readily incorporated into the structure
- One can also regard $G(s)$ as a predictive model. Then, the structure interprets the basic idea of some MPC algorithms

Significance of the Youla parameterization:

1. Most design problems can be formulated in this way: Given a plant $G(s)$, design a controller $C(s)$ so that the feedback system is internally stable and the output $y(s)$ asymptotically tracks a step reference $r(s)$. The procedure is greatly complicated by treating the stability and the performance simultaneously. With the Youla parameterization, one can consider the stability and the performance separately. First, describe all stabilizing $C(s)$s in terms of a stable transfer function $Q(s)$. Then, search the optimal $C(s)$ among the stabilizing controllers. This search is much easier.

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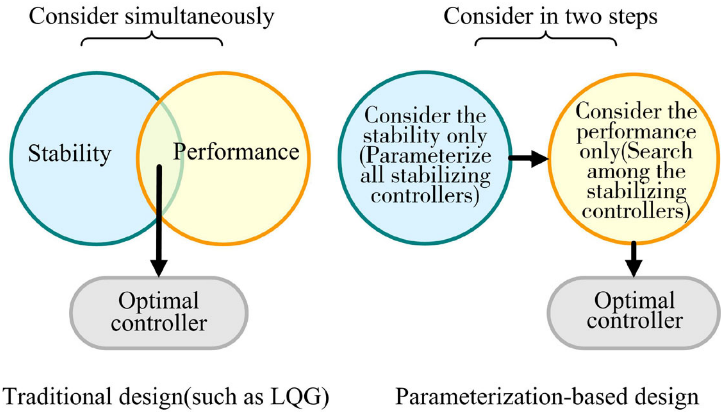

Figure: Two different design procedures

2.The effect of $C(s)$ on $S(s)$ and $T(s)$ is complicated. Nevertheless, $Q(s)$ relates to $S(s)$ and $T(s)$ in a "**linear**" manner, which can significantly simplify the design task for optimal controllers.

$$
\begin{aligned}
S(s) &= \frac{1}{1+G(s)C(s)} = 1 - G(s)Q(s) \\
T(s) &= \frac{G(s)C(s)}{1+G(s)C(s)} = G(s)Q(s)
\end{aligned}
$$

It is evident that $Q(s)$ should be proper, since an improper transfer function is not physically realizable. However, it would be convenient for presentation to temporarily relax the requirement in controller design. When an improper $Q(s)$ is used, the mathematically precise reader should interpret the improper transfer function as a shorthand notation of its proper approximation. e.g., $s+1 \approx (s+1)/(\lambda s+1)$. λ is very small

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Youla Parameterization for Unstable Plants

Write $G(s)$ in the form of its coprime factorization:

$$G(s) = \frac{V(s)}{U(s)}$$

where $V(s)$ and $U(s)$ are stable, proper, real and rational. There exist two stable proper real rational functions $X(s)$ and $Y(s)$ satisfying the equation

$$V(s)X(s) + U(s)Y(s) = 1$$

By checking $\mathbf{H}(s)$, it is known that $C(s) = X(s)/Y(s)$ is a stabilizing controller:

$$\mathbf{H}(s) = \begin{bmatrix} V(s)X(s) & V(s)Y(s) \\ X(s)U(s) & -V(s)X(s) \end{bmatrix}$$

Notice that

$$V(s)\underbrace{[X(s)+U(s)Q(s)]}_{Stable}+U(s)\underbrace{[Y(s)-V(s)Q(s)]}_{Stable}=1$$

for any stable $Q(s)$. It turns out that all controllers for which the feedback system is internally stable can be expressed as

$$C(s)=\frac{X(s)+U(s)Q(s)}{Y(s)-V(s)Q(s)}$$

where $Q(s)$ is any stable transfer function.When $G(s)$ is stable

$$V(s)=G(s),\ U(s)=1,\ X(s)=0,\ Y(s)=1$$

It is an open question to compute the coprime factorization with analytical methods. In latter chapters, an alternative parameterization will be developed.

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3.4 Robust Stability and Robust Performance

Uncertainty Description

Uncertainty: $\begin{cases} \text{Structured uncertainty} \\ \text{Unstructured uncertainty} \end{cases}$

Structured uncertainty: A frequently used structured uncertainty is the parameter uncertainty:

$$G(s) = \frac{a}{s^2 + 2s + 1}, a_{\min} \le a \le a_{max}$$

For rational systems and some specific parameter uncertainty there is a famous test for robust stability

Consider the following characteristic equation:

$$s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0 = 0$$

where n is a nonnegative integer. The uncertain parameters are

$$\alpha_i \leq a_i \leq \beta_i, \ i = 0, 1, ..., n-1$$

To ascertain the stability of the system, one might have to to investigate all possible combinations of parameters. Fortunately, it is possible to investigate only four polynomials

Theorem

The closed-loop system is robust stable if and only if the following four polynomials are stable:

$$\begin{aligned}
q_1(s) &= \alpha_0 + \alpha_1 s + \beta_2 s^2 + \beta_3 s^3 + \alpha_4 s^4 + \alpha_5 s^5 + ..., \\
q_2(s) &= \alpha_0 + \beta_1 s + \beta_2 s^2 + \alpha_3 s^3 + \alpha_4 s^4 + \alpha_5 s^5 + ..., \\
q_3(s) &= \beta_0 + \beta_1 s + \alpha_2 s^2 + \alpha_3 s^3 + \beta_4 s^4 + \beta_5 s^5 + ... \\
q_4(s) &= \beta_0 + \alpha_1 s + \alpha_2 s^2 + \beta_3 s^3 + \beta_4 s^4 + \alpha_5 s^5 + ...
\end{aligned}$$

where n is a nonnegative integer. The uncertain parameters are

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$$q_4(s) = \beta_0 + \alpha_1 s + \alpha_2 s^2 + \beta_3 s^3 + \beta_4 s^4 + \alpha_5 s^5 + ...$$

Example

This example illustrates how to construct the four polynomials. The characteristic equation of a third-order system is

$$s^3 + a_2 s^2 + a_1 s + a_0 = 0.$$

The four polynomials are

$$\begin{aligned} q_1(s) &= s^3 + \beta_2 s^2 + \alpha_1 s + \alpha_0, \\ q_2(s) &= s^3 + \beta_2 s^2 + \beta_1 s + \alpha_0, \\ q_3(s) &= s^3 + \alpha_2 s^2 + \beta_1 s + \beta_0, \\ q_4(s) &= s^3 + \alpha_2 s^2 + \alpha_1 s + \beta_0. \end{aligned}$$

Each polynomial represents a worst case. In such a way one can easily test the stability of a rational system with parameter uncertainty

Unstructured uncertainty: Assume that the nominal plant $G(s)$ and the uncertain plant $\widetilde{G}(s)$ have the same number of RHP poles. The unstructured uncertainty can be written in the form of multiplicative uncertainty:

$$\widetilde{G}(s) = G(s)\left[1 + \delta_m(s)\right], \ \delta_m(s) = \Delta(s)\Delta_m(s)$$

Here $\Delta_m(s)$ is a fixed stable transfer function. $\Delta(s)$ is a variable stable transfer function satisfying $\|\Delta(s)\|_\infty \leq 1$. It can be viewed as a normalized uncertainty. For each frequency ω,

$$|\delta_m(j\omega)| \leq |\Delta_m(j\omega)|$$

$|\Delta_m(j\omega)|$ provides the uncertainty profile
The unstructured uncertainty can be described as a disk in the complex plane: At each frequency ω, the point $\widetilde{G}(j\omega)$ lies in the disk with center $G(j\omega)$ and radius $|\Delta_m(j\omega)|$

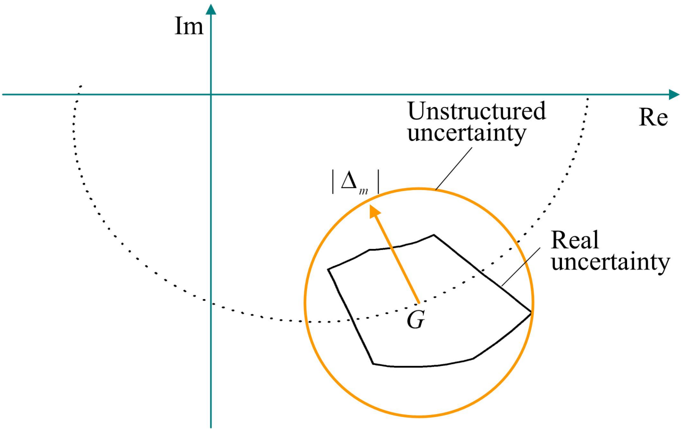

Figure: Disk for unstructured uncertainty

The unstructured uncertainty is important for two reasons:

1. All models used in feedback design should include some unstructured uncertainties to cover unmodeled dynamics, particularly at high frequencies.
2. With the unstructured uncertainty, simple and general results can be obtained for not only robust stability but also robust performance. The price is that the description might be conservative

As an example, the region with an irregular shape in the last figure represents the parameter uncertainty, which is mathematically difficult to deal with

Robust Stability

Robust stability: The internal stability holds for all plants in the uncertain model family

Theorem

Assume that the nominal closed-loop system is internally stable. $C(s)$ provides robust stability if and only if

$$\|\Delta_m(s)T(s)\|_\infty < 1$$

Proof.

By assumption, the nominal feedback system is internally stable. From the Nyquist criterion it is known that for every frequency ω

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Proof (ctd.1).

the Nyquist plot of $L(j\omega) = G(j\omega)C(j\omega)$ does not pass through the point (-1, 0),and its number of counterclockwise encirclements equals the number of poles of $G(s)$ in the RHP plus the number of poles of $C(s)$ in the RHP. It is also known that $G(s)$ and $\widetilde{G}(s)$ have the same number of RHP poles. Thus, $C(s)$ provides robust stability if and only if the Nyquist band of $\widetilde{L}(j\omega) = \widetilde{G}(j\omega)C(j\omega)$ does not include the point (-1, 0) for every frequency ω.

Consider the simple geometric argument in Figure. For a frequency ω, $\widetilde{L}(j\omega)$ is a disk in the complex plane. The disk has the center $L(j\omega)$ and the radius $|\Delta_m(j\omega)G(j\omega)C(j\omega)|$. The radius denotes the maximum perturbed scope of $L(j\omega)$. The distance from $L(j\omega)$ to the point (-1, 0) is $|1 + G(j\omega)C(j\omega)|$. It is clear that $\widetilde{L}(j\omega)$ does not include the point (-1, 0) for every frequency ω if and only if

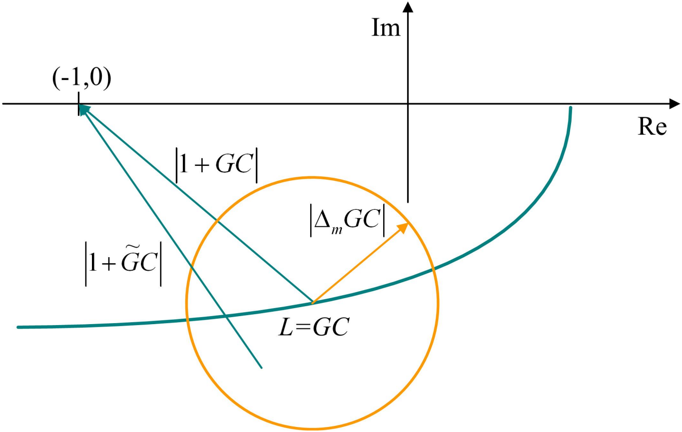

Figure: Graphical interpretation for robust stability.

Proof (ctd.2).

$$|1 + G(j\omega)C(j\omega)| > |\Delta_m(j\omega)G(j\omega)C(j\omega)|.$$

As

$$T(j\omega) = \frac{G(j\omega)C(j\omega)}{1 + G(j\omega)C(j\omega)},$$

the foregoing inequality is equivalent to

$$|\Delta_m(j\omega)T(j\omega)| < 1, \forall \omega.$$

This completes the proof. □

Robust Performance

Robust Performance: The internal stability and performance hold for all plants in the family of uncertain models
When the nominal feedback system is internally stable, the performance requirement is $\|W(s)S(s)\|_\infty < 1$. When there exists uncertainty, $S(s)$ is perturbed to

$$\begin{aligned}\widetilde{S}(s) &= \frac{1}{1+[1+\delta_m(s)]\,G(s)C(s)} \\ &= \frac{S(s)}{1+\delta_m(s)T(s)}.\end{aligned}$$

The robust performance condition should be

$$\|W(s)\widetilde{S}(s)\|_\infty < 1$$

Theorem

Assume that the nominal closed-loop system is internally stable. A sufficient and necessary condition for robust performance is

$$|W(j\omega)S(j\omega)| + |\Delta_m(j\omega)T(j\omega)| < 1, \forall\omega$$

Proof.

From the last figure, the smallest distance form the point (-1, 0) to the point in the disk is $|1 + G(j\omega)C(j\omega)| - |\Delta_m(j\omega)G(j\omega)C(j\omega)|$ for every frequency ω. Then

$$|1 + \widetilde{G}(j\omega)C(j\omega)| \geq |1 + G(j\omega)C(j\omega)| - |\Delta_m(j\omega)G(j\omega)C(j\omega)|$$

By taking the inverse of both sides of the foregoing inequality, it is easy to obtain that

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Proof (ctd.1).

$$|\widetilde{S}(j\omega)| \leq \frac{|S(j\omega)|}{1 - |\Delta_m(j\omega)T(j\omega)|}.$$

The robust performance requires

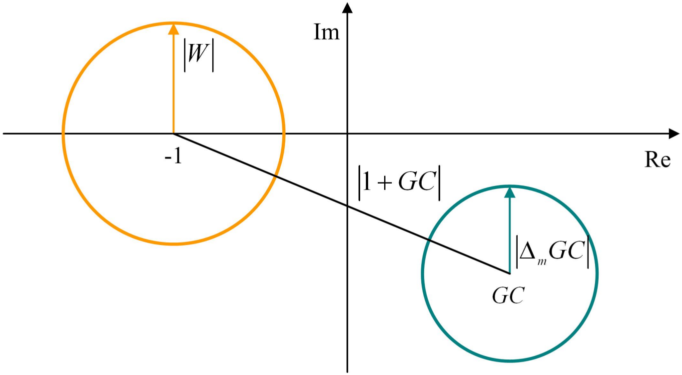
$$\|W(s)\widetilde{S}(s)\|_\infty < 1.$$

The condition holds if and only if the following inequality holds for every ω:

$$|W(j\omega)S(j\omega)| + |\Delta_m(j\omega)T(j\omega)| < 1$$

$\square$

Graphical interpretation for robust performance



When both sides of the test condition are multipled by $|1 + G(j\omega)C(j\omega)|$, two disks are constructed for every frequency ω: One with center -1 and radius $|W(j\omega)|$; the other with center $G(j\omega)C(j\omega)$ and radius $|\Delta_m(j\omega)G(j\omega)C(j\omega)|$. The test condition holds if and only if these two disks are disjoint

Note that robust performance implies both robust stability and nominal performance. It is desirable to make $\|W(s)S(s)\|_\infty$ small for good nominal performance and at the same time make $\|\Delta_m(s)T(s)\|_\infty$ small for good robust stability. Unfortunately, the interdependence of $S(s) + T(s) = 1$ makes the objective a challenge. Improving the nominal performance worsens the robust stability and pushes the system close to instability. Conversely, good robust stability may be obtained by sacrificing the nominal performance

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3.5 Robustness of Systems with Time Delays

The first-Order Plant with Time Delay

This section discusses how the **parameter uncertainty** of the first-order plant with time delay affects the stability and performance of the closed-loop system. The first-order plant with time delay is chosen because

1. This model is widely used in practice, especially in industry.
2. It is easy to deal with the first-order plant and the result can provide good understanding for complex systems.

Assume that the following model has been obtained from experimental data:

$$\widetilde{G}(s) = \frac{\widetilde{K} e^{-\widetilde{\theta} s}}{\widetilde{\tau} s + 1}$$

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$$\widetilde{G}(s) = \frac{\widetilde{K} e^{-\widetilde{\theta} s}}{\widetilde{\tau} s + 1}$$

where
$\widetilde{K}$—Gain
$\widetilde{\tau}$—Time constant
$\widetilde{\theta}$—Time delay
and

$$\begin{aligned}
\widetilde{K} &\in [\widetilde{K}_{min}, \widetilde{K}_{max}], \\
\widetilde{\tau} &\in [\widetilde{\tau}_{min}, \widetilde{\tau}_{max}], \\
\widetilde{\theta} &\in [\widetilde{\theta}_{min}, \widetilde{\theta}_{max}].
\end{aligned}$$

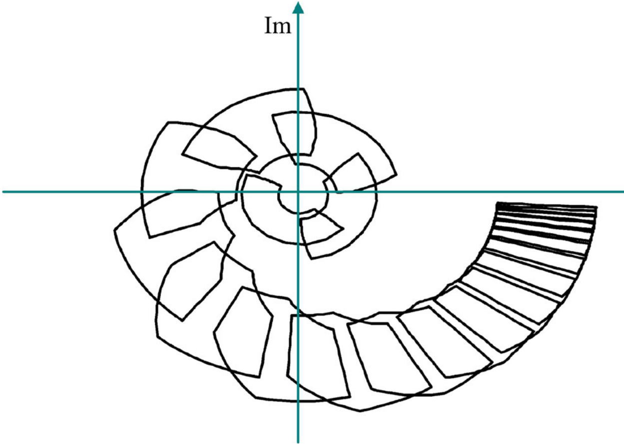

Figure: Model family

Design Method

According to the design requirement, design methods can roughly be classified into two kinds:

1. The design requirement involving the specification about robustness is given for the nominal system. The controller is designed for the nominal plant, and then used for the real plant.
2. The design requirement is proposed for the uncertain system. The controller is designed based on the nominal plant and the associated uncertainty, and then used for the real plant.

In both of the two methods, a nominal plant is needed. In this example, the "center" of the uncertain plant is chosen as the nominal plant, i.e.

$$G(s) = \frac{Ke^{-\theta s}}{\tau s + 1}$$

where

$$K = \frac{\widetilde{K}_{min} + \widetilde{K}_{max}}{2}, \tau = \frac{\widetilde{\tau}_{min} + \widetilde{\tau}_{max}}{2}, \theta = \frac{\widetilde{\theta}_{min} + \widetilde{\theta}_{max}}{2}$$

The parameter uncertainty is

$$\begin{aligned} |\delta K| \le \Delta K &= |\widetilde{K}_{max} - K| < |K|, \\ |\delta \tau| \le \Delta \tau &= |\widetilde{\tau}_{max} - \tau| < |\tau|, \\ |\delta \theta| \le \Delta \theta &= |\widetilde{\theta}_{max} - \theta| < |\theta|. \end{aligned}$$

Then the uncertain model family can be written as:

$$\widetilde{G}(s) = \frac{(K + \delta K)e^{-(\theta + \delta\theta)s}}{(\tau + \delta\tau)s + 1}$$

Assume that there are $\pm 20\%$ or $\pm 40\%$ perturbation for gain, time constant, and time delay, respectively. Typical responses of the closed-loop system are shown in Figures. It is observed that

1. With the gain increasing, the rise time decreases and the overshoot increases. Contrarily, with the gain decreasing, the rise time increases and the overshoot decreases or disappears.
2. With the time constant increasing, the rise time increases and the response tends to be steady. If the time constant decreases, the rise time decreases and the response begins to oscillate.
3. With the time delay increasing, the overshoot increases. The overshoot decreases or disappears when the time delay decreases.

It is noticed that when the time constant decreases and the gain and the time delay increase, the change of the closed-loop response is the largest (**the worst case**)

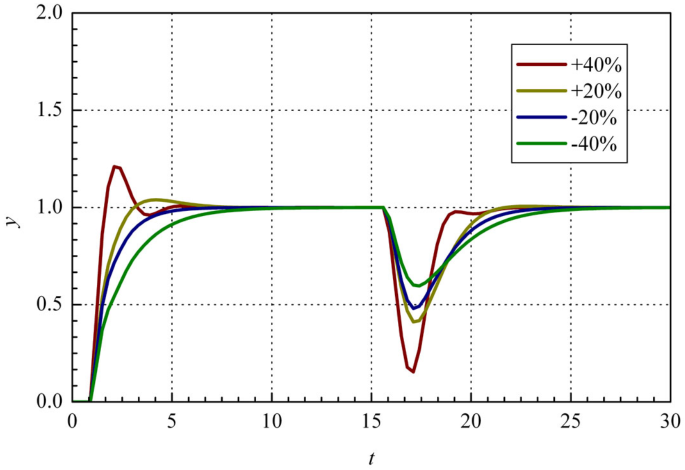

Figure: Effect of the gain uncertainty

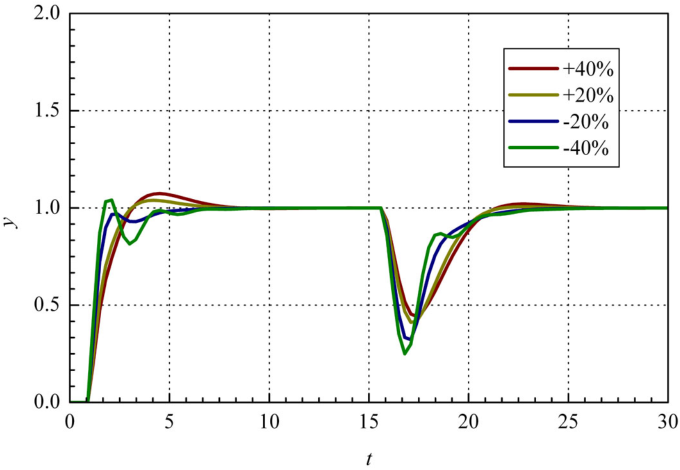

Figure: Effect of the time constant uncertainty

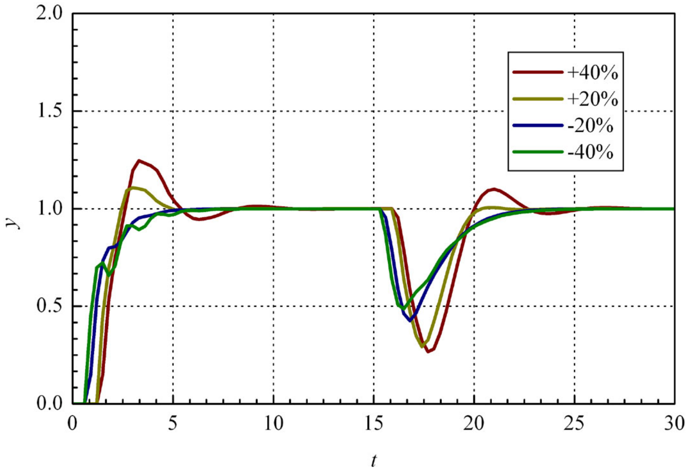

Figure: Effect of the time delay uncertainty

Parameter Uncertainty => Unstructured uncertainty

Theoretically, the effect of the uncertainty on the stability and performance of the closed-loop system can be analyzed by employing the condition derived in the last section. To use the result, one has to convert the parameter uncertainty of into the unstructured uncertainty

Rewrite the uncertain model in the form of

$$\widetilde{G}(s) = \frac{Ke^{-\theta s}}{\tau s + 1}[1 + \delta_m(s)]$$

where

$$\delta_m(s) = \frac{K + \delta K}{K} \frac{\tau s + 1}{(\tau + \delta\tau)s + 1} e^{-\delta\theta s} - 1$$

Parameter Uncertainty => Unstructured uncertainty

Theoretically, the effect of the uncertainty on the stability and performance of the closed-loop system can be analyzed by employing the condition derived in the last section. To use the result, one has to convert the parameter uncertainty of into the unstructured uncertainty

Rewrite the uncertain model in the form of

$$\widetilde{G}(s) = \frac{Ke^{-\theta s}}{\tau s + 1}[1 + \delta_m(s)]$$

where

$$\delta_m(s) = \frac{K + \delta K}{K} \frac{\tau s + 1}{(\tau + \delta\tau)s + 1} e^{-\delta\theta s} - 1$$

Let $|\Delta_m(j\omega)|$ be the bound of $|\delta_m(j\omega)|$, namely $|\delta_m(j\omega)| \le |\Delta_m(j\omega)|$. $|\Delta_m(j\omega)|$ is equal to the radius of the smallest disk containing the parameter uncertainty boundary.

When there are simultaneous uncertainties on the gain, the time constant, and the time delay, the following analytical expression for the uncertainty profile is obtained:

$$|\Delta_m(j\omega)| = \begin{cases} \left| \dfrac{|K| + \Delta K}{|K|} \dfrac{j\tau\omega + 1}{j(\tau - \Delta\tau)\omega + 1} e^{j\Delta\theta\omega} - 1 \right|, & \omega < \omega^* \\ \left| \dfrac{|K| + \Delta K}{|K|} \dfrac{j\tau\omega + 1}{j(\tau - \Delta\tau)\omega + 1} \right| + 1, & \omega \ge \omega^* \end{cases}$$

where ω^* is determined by

$$\Delta\theta\omega^* + \arctan \frac{\Delta\tau\omega^*}{1 + \tau(\tau - \Delta\tau)\omega^{*2}} = \pi, \frac{\pi}{2} \le \Delta\theta\omega^* \le \pi$$

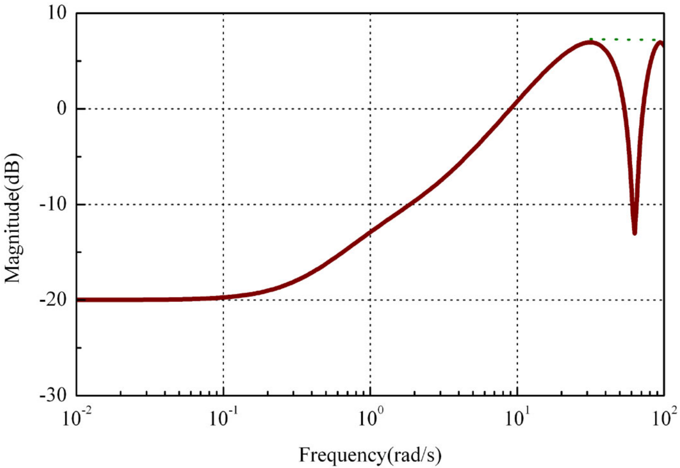

Figure: Unstructured uncertainty profile

In particular, when only the gain is uncertain, that is, $\Delta\tau = \Delta\theta = 0$, the expression simplifies to

$$|\Delta_m(j\omega)| = \Delta K/|K|$$

When only the time constant is uncertain, that is, $\Delta K = \Delta\theta = 0$, the expression simplifies to

$$|\Delta_m(j\omega)| = \left|\frac{j\tau\omega + 1}{j(\tau - \Delta\tau)\omega + 1} - 1\right|$$

When only the time delay is uncertain, $\Delta\tau = \Delta K = 0$ and $\omega^* = \pi/\Delta\theta$. In this case,

$$|\Delta_m(j\omega)| = \begin{cases} |e^{j\Delta\theta\omega} - 1|, & \omega < \pi/\Delta\theta \\ 2, & \omega \geq \pi/\Delta\theta \end{cases}$$

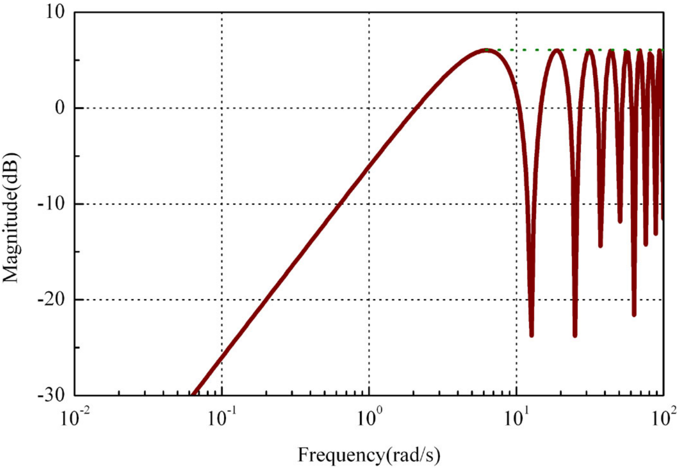

Figure: Uncertainty profile for time delay uncertainty

Brief introduction about the proof: For all δK, $\delta\tau$, and $\delta\theta$, $|\Delta_m(j\omega)|$ is the minimum upper bound that makes $|\delta_m(j\omega)| \le |\Delta_m(j\omega)|$. If the maximum distance from (1,0) to $\delta_m(s)+1$ is determined, then the distance is $|\Delta_m(j\omega)|$. For $|\delta K| \le \Delta K$, $|\delta|\tau \le \Delta\tau$, $|\delta\theta| \le \Delta\theta$, and some frequency $s = j\omega$, all possible points of $\delta_m(j\omega)+1$ are located in a certain region. The geometrical relationship shows that the point $\delta K = \Delta K, \delta\tau = -\Delta\tau$, and $\delta\theta = \Delta\theta$ is the farthest one from (1,0) for all $\omega < \omega^*$, where ω^* is the frequency at which the angle of this point equals to π. Furthermore, the result for $\omega \ge \omega^*$ can be derived by utilizing the triangular inequality

An engineering method

In practice, it is not convenient to test the robust performance by the sufficient and necessary condition given in the last section. A simple engineering method is to examine whether the internal stability and performance hold for the **worst case**

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End of Chapter 3