Chapter 10 Analysis of MIMO Systems

10.1 Zeros and Poles of a MIMO Plant

Causality: An $n \times n$ MIMO plant $\mathbf{G}(t)$ is causal if all of its elements $G_{ij}(t)(i = 1, 2, ..., n; j = 1, 2, ..., n)$ are causal. Such a MIMO plant can be described by a square transfer function matrix $\mathbf{G}(s)$, whose elements are in the form of proper transfer functions

Proper: A transfer function matrix $\mathbf{G}(s)$ is proper if all its elements $G_{ij}(s)$ are proper

Strictly proper: $\mathbf{G}(s)$ is strictly proper if all its elements $G_{ij}(s)$ are strictly proper

Semi-proper: $\mathbf{G}(s)$ is semi-proper if $\mathbf{G}(s)$ is proper but not strictly proper. For scalar plants, "semi-proper" is equivalent to "bi-proper"

Improper: All transfer function matrices that are not proper are improper

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Strictly proper: $\mathbf{G}(s)$ is strictly proper if all its elements $G_{ij}(s)$ are strictly proper
Semi-proper: $\mathbf{G}(s)$ is semi-proper if $\mathbf{G}(s)$ is proper but not strictly proper. For scalar plants, "semi-proper" is equivalent to "bi-proper"
Improper: All transfer function matrices that are not proper are improper

Pole polynomial: The pole polynomial $\pi(s)$ of **G(s)** is the least common denominator of all non-identically-zero minors of **G(s)**
Pole: The pole is the root of the equation $\pi(s) = 0$

Stability: A system is stable if and only if all its poles are in the open LHP

Example

Consider the following plant:

$$\mathbf{G}(s) = \begin{bmatrix} \frac{1}{s+3} & \frac{1}{s-2} \\ \frac{3}{s+3} & \frac{s+1}{s-2} \end{bmatrix}$$

The minors of order 1 are the determinant of the elements of $\mathbf{G}(s)$. The minor of order 2 is the determinant of the plant itself

Example (ctd.1)

According to the minors of all orders, the least common denominator is

$$\pi(s) = (s+3)(s-2)$$

Therefore, $\mathbf{G}(s)$ has two poles: One is at $s = -3$ and the other is at $s = 2$

Let "det" denote determinant

Zero polynomial: The zero polynomial $\zeta(s)$ is the numerator of $\det[\mathbf{G}(s)]$, **provided** that $\det[\mathbf{G}(s)]$ has been adjusted to have the pole polynomial as its denominator

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If a point is a zero of $\mathbf{G}(s)$, the rank of $\mathbf{G}(s)$ at this point is less than its normal rank

Normal rank: The rank of $\mathbf{G}(s)$ for every s in the set of complex numbers, except for a finite number of points

Example

Consider the plant in Example 10.1.1. Adjust $\det \mathbf{G}(s)$ so that its denominator is the pole polynomial:

$$\det[\mathbf{G}(s)] = \frac{s-2}{(s+3)(s-2)}$$

Then

$$\zeta(s) = (s-2)$$

Hence, $\mathbf{G}(s)$ has one zero at $s = 2$

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Simple zero: When $\mathbf{G(s)}$ has only one pole at $s = p_0$, p_0 is said to be a simple pole

Multiple zero: If $\mathbf{G(s)}$ has multiple poles at $s = p_0$, p_0 is a multiple pole

Features of MIMO zeros:

- MIMO plants can have zeros and poles at the same location. In general, it is **impossible** to find all the zeros of a plant from the condition $\det[\mathbf{G}(s)] = 0$. When forming the determinant, zeros and poles at the same location cancel
- The zero location of a MIMO system is no longer related to the zero location of the individual SISO transfer function constituting the MIMO system. Thus, it is possible for a MIMO system to be NMP even though all the SISO transfer functions are MP

NMP: A plant **G(s)** is NMP if its transfer function matrix contains zeros in the closed RHP or contains a time delay
MP: Otherwise, the plant is MP

With regard to MIMO plants, different definitions of the time delay will result in different MP plants. This book gives a definition for decoupling control in Chapter 13

Example

The plant

$$\mathbf{G}(s) = \frac{1}{s+1}\begin{bmatrix} s+3 & 2 \\ 3 & 1 \end{bmatrix}$$

has a zero at $s = 3$ and thus is NMP. However, all the SISO transfer functions are MP.

Example (ctd.1)

The normal rank of $\mathbf{G}(s)$ is 2. Substituting $s = 3$ into $\mathbf{G}(s)$ yields

$$\mathbf{G}(3) = \frac{1}{4}\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}$$

It can be seen the feature of the resulting matrix is that its two rows are not independent. The rank of $\mathbf{G}(3)$ is 1, which is less than the normal rank of $\mathbf{G}(s)$

The multiplicity of a zero is closely related to the rank of the plant at the zero

Theorem

If the rank of $\mathbf{G}(z)$ is k, then $\mathbf{G}(s)$ has a zero at $s = z$ of multiplicity at least $n - k$.

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Theorem

If the rank of $\mathbf{G}(z)$ is k, then $\mathbf{G}(s)$ has a zero at $s = z$ of multiplicity at least $n - k$.

Proof.

Since rank$[\mathbf{G}(z)] = k$, with proper elementary transformations $\mathbf{G}(s)$ can be transformed into the following form:

$$\mathbf{G_1}(s) = \begin{bmatrix} b_1(s) \\ \vdots \\ b_k(s) \\ b_{k+1}(s) \\ \vdots \\ b_n(s) \end{bmatrix}$$

where $b_1(z), b_2(z), ..., b_k(z)$ are linearly independent rows. $b_{k+i}(z)(i = 1, 2, ..., n-k)$ either are zero or can be written as

$$b_{k+i}(z) = \alpha_{i1}b_1(z) + \alpha_{i2}b_2(z) + ... + \alpha_{ik}b_k(z)$$

where $\alpha_{i1}, \alpha_{i2}, ..., \alpha_{ik}$ are constants that are not all zero.

Proof.

Carry out elementary transformations with regard to $\mathbf{G_1}(s)$, so that

$$\mathbf{G_2}(s) = \begin{bmatrix} b_1(s) \\ \vdots \\ b_k(s) \\ b'_{k+1}(s) \\ \vdots \\ b'_n(s) \end{bmatrix}$$

where $b'_{k+i}(s) = b_{k+i}(s) - \alpha_{i1}b_1(s) - \alpha_{i2}b_2(s) - \ldots - \alpha_{ik}b_k(s)$ if $b_{k+i}(z) \neq 0$ and $b'_{k+i}(s) = b_{k+i}(s)$ if $b_{k+i}(z) = 0$. Then $b'_{k+i}(s)(i = 1, 2, \ldots, n-k)$ must contain the factor $s - z$.
As elementary transformations do not change the value of determinant, $\mathbf{G}(s)$ has a zero at z of multiplicity at least $n-k$ □

10.2 Singular Values

Consider a fixed frequency ω. $T(j\omega)$ is a **complex number**

Gain in SISO systems: For a SISO system, $y(s) = T(s)r(s)$. The gain at a given frequency can be simply expressed as

$$\frac{|y(j\omega)|}{|r(j\omega)|} = \frac{|T(j\omega)r(j\omega)|}{|r(j\omega)|} = |T(j\omega)|$$

The gain depends on the frequency, but is independent of the input magnitude

Gain in MIMO systems: In an $n \times n$ system $\mathbf{T}(s)$, the case is different because both the input $\mathbf{r}(s)$ and the output $\mathbf{y}(s) = \mathbf{T}(s)\mathbf{r}(s)$ are **vectors**. To investigate the gain at a given frequency, we need to "sum up" the magnitudes of input and output signals in each vector by utilizing norms

If the 2-norm, a frequently used measure, is selected, the "size" of the input signal is

$$\|\mathbf{r}(j\omega)\|_2 := \left[\sum_{j=1}^{n} |r_j(j\omega)|^2\right]^{1/2}$$

and the "size" of the output signal is

$$\|\mathbf{y}(j\omega)\|_2 := \left[\sum_{i=1}^{n} |y_i(j\omega)|^2\right]^{1/2}$$

The gain of the system at a given frequency can then be described by the ratio

$$\frac{\|\mathbf{y}(j\omega)\|_2}{\|\mathbf{r}(j\omega)\|_2} = \frac{\|\mathbf{T}(j\omega)\mathbf{r}(j\omega)\|_2}{\|\mathbf{r}(j\omega)\|_2} = \left[\frac{\sum_i |y_i(j\omega)|^2}{\sum_j |r_j(j\omega)|^2}\right]^{1/2}$$

In addition to the frequency, the gain also depends on the input direction

Example

Consider a 2×2 system

$$\mathbf{T} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

The inputs are

$$\mathbf{r_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{r_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{r_3} = \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix},$$
$$\mathbf{r_4} = \begin{bmatrix} 0.707 \\ -0.707 \end{bmatrix}, \mathbf{r_5} = \begin{bmatrix} 0.8 \\ -0.6 \end{bmatrix}$$

These inputs have the same magnitude $\|\mathbf{r}\|_2 = 1$, but they are in different directions

Example (ctd.1)

Compute the system output for the five inputs:

$$\mathbf{y_1} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \mathbf{y_2} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \mathbf{y_3} = \begin{bmatrix} 2.12 \\ 4.95 \end{bmatrix}$$
$$\mathbf{y_4} = \begin{bmatrix} -0.707 \\ -0.707 \end{bmatrix}, \mathbf{y_5} = \begin{bmatrix} -0.40 \\ 0 \end{bmatrix}$$

The 2-norms of these outputs are

$$\|\mathbf{y_1}\|_2 = 3.16, \|\mathbf{y_2}\|_2 = 4.47, \|\mathbf{y_3}\|_2 = 5.38,$$
$$\|\mathbf{y_4}\|_2 = 1.00, \|\mathbf{y_5}\|_2 = 0.40$$

It is observed that the inputs having the same magnitude and different directions relate to different system gains.

As it is known, the eigenvalues of a MIMO system reflect its gain characteristic. Let $\lambda_{ei}[\mathbf{T}(j\omega)](i = 1, 2, ..., n)$ denote the eigenvalues of $\mathbf{T}(j\omega)$. The sum of the eigenvalues of $\mathbf{T}(j\omega)$ is equal to the trace of $\mathbf{T}(j\omega)$ (that is, the sum of the diagonal elements):

$$\text{Trace}[\mathbf{T}(j\omega)] = \sum_i \lambda_{ei}[\mathbf{T}(j\omega)]$$

The largest eigenvalue is called **spectral radius**, which is denoted by

$$\rho[\mathbf{T}(j\omega)] = \max_i |\lambda_{ei}[\mathbf{T}(j\omega)]|$$

Let the complex number t_{ij} be the elements of $\mathbf{T}(j\omega)$. Define the norm of a complex matrix as

$$\|\mathbf{T}(j\omega)\|_\infty = \max_i \sum_j |t_{ij}|$$

If $\lambda_{ei}[\mathbf{T}(j\omega)]$ is an eigenvalue of $\mathbf{T}(j\omega)$ and $\mathbf{r}(j\omega)$ is a corresponding eigenvector, then

$$|\mathbf{T}(j\omega)\mathbf{r}(j\omega)| = |\lambda_{ei}[\mathbf{T}(j\omega)]||\mathbf{r}(j\omega)|$$

As

$$|\mathbf{T}(j\omega)\mathbf{r}(j\omega)| \leq \|\mathbf{T}(j\omega)\|_\infty |\mathbf{r}(j\omega)|$$

we have

$$\rho[\mathbf{T}(j\omega)] \leq \|\mathbf{T}(j\omega)\|_\infty$$

The eigenvalues of a system, however, do not provide a useful means for generalizing the SISO gain, because they only measure the gain in the special case where the input and the output are in the same direction

Example

Consider the system $\mathbf{y} = \mathbf{T}\mathbf{r}$ with

$$\mathbf{T} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

There are two eigenvalues

$$\lambda_{e1}[\mathbf{T}] = \lambda_{e2}[\mathbf{T}] = 0$$

To conclude that the gain of the system is zero is clearly misleading. For example, with the input

$$\mathbf{r} = [0\ 1]^T$$

we have the output

$$\mathbf{y} = [1\ 0]^T$$

A good measure for the MIMO gain at a given frequency ω is the singular value. The singular values of a complex matrix $\mathbf{T}(j\omega)$, denoted by $\sigma_i[\mathbf{T}(j\omega)]$, are the n square roots of the eigenvalues of $\mathbf{T}^H(j\omega)\mathbf{T}(j\omega)$, that is,

$$\sigma_i[\mathbf{T}(j\omega)] \;=\; \left\{\lambda_{ei}[\mathbf{T}^H(j\omega)\mathbf{T}(j\omega)]\right\}^{1/2}, i = 1, 2, ..., n$$

where the superscript H denotes the complex conjugate transpose of a matrix: $\mathbf{T}^H(j\omega) = \bar{\mathbf{T}}^T(j\omega)$

For convenience, the ordering $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_n$ is adopted. In general, the singular values must be computed numerically. However, for 2×2 matrices analytical expressions can be obtained.

Instead of a single gain, there are **a group of gains** in MIMO systems

The largest gain for all input directions is equal to the maximum singular value:

$$\bar{\sigma}[\mathbf{T}(j\omega)] = \max_{\mathbf{r}(j\omega)\neq 0} \frac{\|\mathbf{T}(j\omega)\mathbf{r}(j\omega)\|_2}{\|\mathbf{r}(j\omega)\|_2}$$

and the smallest gain for all input directions is equal to the minimum singular value:

$$\underline{\sigma}[\mathbf{T}(j\omega)] = \min_{\mathbf{r}(j\omega)\neq 0} \frac{\|\mathbf{T}(j\omega)\mathbf{r}(j\omega)\|_2}{\|\mathbf{r}(j\omega)\|_2}$$

A convenient way of representing a matrix that exposes its internal structure is to use Singular Value Decomposition (SVD). The SVD of $\mathbf{T}(j\omega)$ is given by

$$\begin{aligned}\mathbf{T}(j\omega) &= \mathbf{U}(j\omega)\mathbf{\Sigma}(j\omega)\mathbf{V}^{\mathbf{H}}(j\omega)\\ &= \sum_{i=1}^{n} \sigma_i[\mathbf{T}(j\omega)]\mathbf{u_i}(j\omega)\mathbf{v_i}^H(j\omega)\end{aligned}$$

where

$$\mathbf{U}(j\omega) = [\ \mathbf{u_1}(j\omega) \quad \mathbf{u_2}(j\omega) \quad \dots \quad \mathbf{u_n}(j\omega)\ ]$$

and $\mathbf{U}^H(j\omega) = \mathbf{U}^{-1}(j\omega)$,

$$\mathbf{V}(j\omega) = [\ \mathbf{v_1}(j\omega) \quad \mathbf{v_2}(j\omega) \quad \dots \quad \mathbf{v_n}(j\omega)\ ]$$

and $\mathbf{V}^H(j\omega) = \mathbf{V}^{-1}(j\omega)$. $\mathbf{\Sigma}(j\omega)$ can be written as

$$\mathbf{\Sigma}(j\omega) = \mathrm{diag}\{\sigma_1[\mathbf{T}(j\omega)], \sigma_2[\mathbf{T}(j\omega)], \dots, \sigma_n[\mathbf{T}(j\omega)]\}$$

Here "diag" denotes a diagonal matrix

The columns of $\mathbf{V}(j\omega)$ and $\mathbf{U}(j\omega)$ are unit eigenvectors of $\mathbf{T}^H(j\omega)\mathbf{T}(j\omega)$ and $\mathbf{T}(j\omega)\mathbf{T}^H(j\omega)$, respectively. They are known as the right singular vectors and the left singular vectors of the matrix. By SVD an arbitrary matrix can be decomposed into a "rotation" ($\mathbf{V}^H(j\omega)$) followed by scaling ($\mathbf{\Sigma}(j\omega)$) followed by a "rotation" ($\mathbf{U}(j\omega)$)

Example

Consider again the system

$$\mathbf{T} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

The singular value decomposition is

$$\mathbf{T} = \begin{bmatrix} -0.4046 & -0.9145 \\ -0.9145 & 0.4046 \end{bmatrix} \begin{bmatrix} 5.4650 & 0 \\ 0 & 0.3660 \end{bmatrix} \begin{bmatrix} -0.5760 & 0.8174 \\ -0.8174 & -0.5760 \end{bmatrix}$$

The largest gain of 5.4650 relates to the input in the direction

$$[-0.5760 - 0.8174]^T$$

and the smallest gain of 0.3660 relates to the input in the direction

$$[0.8174 - 0.5760]^T$$

10.3 Norms of Signals and Systems

Last section: The MIMO gain was considered only at individual frequencies

This section:Estimate the MIMO gain for the whole system

Consider the vector function $\mathbf{r}(s)$ of dimension n

2-norm for signals:

$$\|\mathbf{r}(s)\|_2 := \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{r}^H(j\omega)\mathbf{r}(j\omega)d\omega\right]^{1/2}$$

If $\mathbf{r}(s)$ does not have any poles in the closed RHP, Parseval's theorem yields an equivalent time domain expression:

$$\|\mathbf{r}(t)\|_2 = \left[\int_{-\infty}^{\infty} \mathbf{r}^T(t)\mathbf{r}(t)dt\right]^{1/2}$$

Assume that the matrix-valued function $\mathbf{T}(s)$ of dimension $n \times n$ is a strictly proper transfer function matrix without poles on the imaginary axis

2-norm for systems:

$$\|\mathbf{T}(s)\|_2 := \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Trace} \left[\mathbf{T}^H(j\omega)\mathbf{T}(j\omega) \right] d\omega \right\}^{1/2}$$

If $\mathbf{T}(s)$ does not have any poles in the closed RHP, by Parseval's theorem we have

$$\begin{aligned} \|\mathbf{T}(s)\|_2 &= \|\mathbf{T}(t)\|_2 \\ &= \left\{ \int_{-\infty}^{\infty} \text{Trace} \left[\mathbf{T}^T(t)\mathbf{T}(t) \right] dt \right\}^{1/2} \end{aligned}$$

The ∞-norm is sub-multiplicative:

$$\|\mathbf{T}_1(s)\mathbf{T}_2(s)\|_\infty \le \|\mathbf{T}_1(s)\|_\infty \|\mathbf{T}_2(s)\|_\infty$$

For beginners, it is easy to confuse the norm concept in this section with that in the last section, where the norm is defined for vectors and matrices whose elements are **complex numbers**. The norm in this section is defined for vectors and matrices whose elements are **functions**

Consider the linear system

$$\mathbf{y}(s) = \mathbf{T}(s)\mathbf{r}(s)$$

An interesting problem is how to quantify the least upper bound of the system output $y(s)$ for a known input $\mathbf{r}(s)$, or equivalently, how large the system gain is. The system gains are shown in in Table

	$\mathbf{r}(t) = \delta(t)\mathbf{I}$	$\|\mathbf{r}(t)\|_2$
$\|\mathbf{y}(t)\|_2$	$\|\mathbf{T}(s)\|_2$	$\|\mathbf{T}(s)\|_\infty$

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	$\mathbf{r}(t) = \delta(t)\mathbf{I}$	$\Vert\mathbf{r}(t)\Vert_2$
$\Vert\mathbf{y}(t)\Vert_2$	$\Vert\mathbf{T}(s)\Vert_2$	$\Vert\mathbf{T}(s)\Vert_\infty$

Assume that a unit pulse is applied to each input in due order: $\mathbf{r}(t) = \delta(t)\mathbf{I}$. We have

$$\|\mathbf{y}(t)\|_2 = \|\mathbf{T}(t)\|_2 = \|\mathbf{T}(s)\|_2$$

This implies that for this specific input, the least upper bound on the system output is the 2-norm of the system transfer function matrix.

Assume that the input is bounded: $\|\mathbf{r}(s)\|_2 \leq 1$. The ∞-norm of the transfer function matrix equals the maximum energy of the output. This is shown as follows:

$$\begin{aligned}\|\mathbf{y}(s)\|_2^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{r}^H(j\omega)\mathbf{T}^H(j\omega)\mathbf{T}(j\omega)\mathbf{r}(j\omega)d\omega \\ &\leq \sup_{\omega} |\mathbf{T}(j\omega)|^2 \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{r}^H(j\omega)\mathbf{r}(j\omega)d\omega\end{aligned}$$

or

$$\|\mathbf{y}(t)\|_2^2 \le \|\mathbf{T}(s)\|_\infty^2 \|\mathbf{r}(t)\|_2^2$$

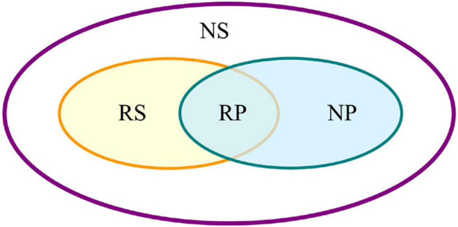
Thus, $\|\mathbf{T}(s)\|_\infty$ is an upper bound of the system output
To prove that it is the least upper bound, it is enough to show that the bound can be reached for a specific input. The specific input is a constructed frequency domain impulse occurring at the frequency where $|\mathbf{T}(j\omega)|$ is maximum. The constructing procedure, which can be found in Section 3.1, will not be repeated here

10.4 Nominal Stability and Performance

In order to work well in a real system, the following objectives have to be met by a controller:

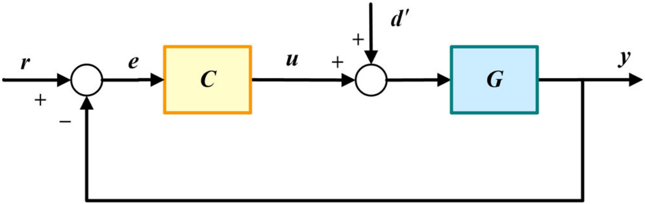
- Nominal stability (NS)
- Nominal performance (NP)
- Robust stability (RS)
- Robust performance (RP)

The nominal stability is mandatory, while the robust performance is normally the ultimate design objective



Nominal stability and nominal performance are addressed in this section. The other two objectives are going to be studied in the next two sections

Consider the control system consisting of an $n \times n$ plant $\mathbf{G}(s)$ and an $n \times n$ controller $\mathbf{C}(s)$. **Assume that there is not any unstable hidden mode in $\mathbf{G}(s)$**. The assumption implies that the plant can be stabilized by using feedback control. It is satisfied if there is not any RHP zero-pole cancellation in $\mathbf{G}(s)$



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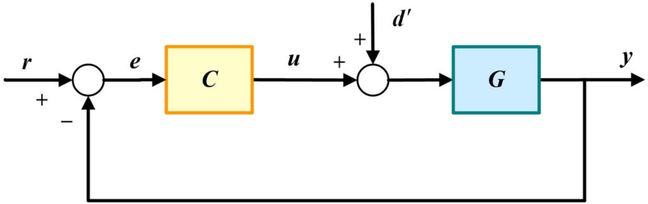


Theorem (MIMO Nyquist Stability Criterion)

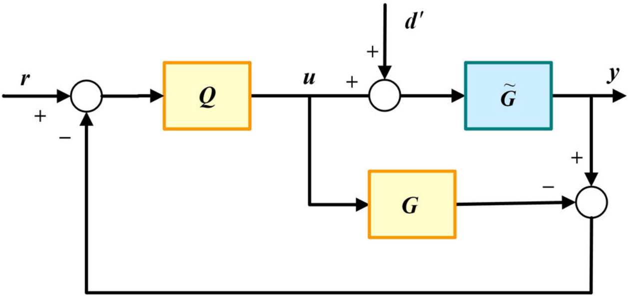
Let n be the number of unstable poles of $\mathbf{L}(s)$. The closed-loop system is stable if and only if the Nyquist plot of $\det[\mathbf{I}+\mathbf{L}(s)]$ does not pass through the origin, and encircles it n times counterclockwise.

The test for internal stability introduced in Chapter 3 is also applicable to MIMO systems. The unity feedback control system is internally stable if and only if all elements in the transfer function matrix $\mathbf{H}(s)$ are stable.

$$\begin{bmatrix} \mathbf{y}(s) \\ \mathbf{u}(s) \end{bmatrix} = \mathbf{H}(s) \begin{bmatrix} \mathbf{r}(s) \\ \mathbf{d}'(s) \end{bmatrix}$$

where

$$\mathbf{H}(s) = \begin{bmatrix} \mathbf{G}(s)\mathbf{C}(s)[\mathbf{I}+\mathbf{G}(s)\mathbf{C}(s)]^{-1} & [\mathbf{I}+\mathbf{G}(s)\mathbf{C}(s)]^{-1}\mathbf{G}(s) \\ \mathbf{C}(s)[\mathbf{I}+\mathbf{G}(s)\mathbf{C}(s)]^{-1} & -\mathbf{C}(s)[\mathbf{I}+\mathbf{G}(s)\mathbf{C}(s)]^{-1}\mathbf{G}(s) \end{bmatrix}$$



The performance analysis for MIMO systems is similar to that for SISO systems. Consider the IMC structure shown in Figure, where $\tilde{\mathbf{G}}(s)$ is the plant and $\mathbf{G}(s)$ is the model. Assume that the model is exact (that is, $\tilde{\mathbf{G}}(s) = \mathbf{G}(s)$). The unit feedback loop controller can be written as

$$\mathbf{C}(s) \quad = \quad \mathbf{Q}(s)[\mathbf{I} - \mathbf{G}(s)\mathbf{Q}(s)]^{-1}$$

Define the sensitivity function as

$$\begin{aligned} \mathbf{S}(s) &= [\mathbf{I} + \mathbf{G}(s)\mathbf{C}(s)]^{-1} \\ &= \mathbf{I} - \mathbf{G}(s)\mathbf{Q}(s) \end{aligned}$$

The complementary sensitivity function is

$$\begin{aligned} \mathbf{T}(s) &= \mathbf{I} - \mathbf{S}(s) \\ &= \mathbf{G}(s)\mathbf{C}(s)[\mathbf{I} + \mathbf{G}(s)\mathbf{C}(s)]^{-1} \\ &= \mathbf{G}(s)\mathbf{Q}(s) \end{aligned}$$

First of all, the steady-state performance of the closed-loop system is characterized. Let m be the largest integer satisfying

$$\mathrm{rank}\{\lim_{s\to 0}[s^m \mathbf{L}(s)]\} = n$$

$\mathbf{L}(s)$ is said to be of Type m. It is seen that $\mathbf{L}(s)$ has at least $n \times m$ poles at the origin

The corresponding sensitivity function matrix satisfies

$$\lim_{s\to 0}[s^{-k}\mathbf{S}(s)] = \mathbf{0}, k = 1, 2, ..., m-1$$

If the closed-loop system is stable, as $t \to \infty$ the closed-loop system perfectly tracks reference changes of the form $\sum_{k=0}^{m} \mathbf{a_k} s^{-k}$, where $\mathbf{a_k}$ are real constant vectors

In particular, a Type 1 system requires

$$\lim_{s\to 0}[\mathbf{G}(s)\mathbf{Q}(s)] = \mathbf{I}$$

and a Type 2 system requires

$$\begin{aligned}\lim_{s\to 0}[\mathbf{G}(s)\mathbf{Q}(s)] &= \mathbf{I} \\ \lim_{s\to 0}\frac{d}{ds}[\mathbf{G}(s)\mathbf{Q}(s)] &= \mathbf{0}\end{aligned}$$

Then, consider the dynamic performance of MIMO systems. Let $\mathbf{W_{p1}}(s)$ and $\mathbf{W_{p2}}(s)$ be two weighting functions. The H_2 optimal control of MIMO systems is defined as

$$\begin{aligned} & \min \|\mathbf{W_{p2}}(s)\mathbf{S}(s)\mathbf{W_{p1}}(s)\|_2^2 \\ = \; & \min \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{Trace}\left\{ \begin{array}{c} [\mathbf{W_{p2}}(j\omega)\mathbf{S}(j\omega)\mathbf{W_{p1}}(j\omega)]^H \cdot \\ [\mathbf{W_{p2}}(j\omega)\mathbf{S}(j\omega)\mathbf{W_{p1}}(j\omega)] \end{array} \right\} d\omega \end{aligned}$$

The index for H_∞ optimal control is expressed as

$$\begin{aligned} & \min \|\mathbf{W_{p2}}(s)\mathbf{S}(s)\mathbf{W_{p1}}(s)\|_\infty \\ = \; & \min \sup_{\omega} \bar{\sigma}\left[\mathbf{W_{p2}}(j\omega)\mathbf{S}(j\omega)\mathbf{W_{p1}}(j\omega)\right] \end{aligned}$$

Generally speaking, $\mathbf{W_{p1}}(s)$ is more important than $\mathbf{W_{p2}}(s)$, because $\mathbf{W_{p1}}(s)$ is needed for all designs, while $\mathbf{W_{p2}}(s)$ is not necessary in some applications

$\mathbf{W_{p1}}(s)$ is the input weighting function. Excite the system in separate experiments with n different linearly independent inputs $\mathbf{r_i}(s)(i = 1, 2, ..., n)$. For one experiment the error is $\mathbf{e_i}(s) = \mathbf{S}(s)\mathbf{r_i}(s)$. Define $\mathbf{W_{p1}}(s) = [\mathbf{r_1}(s), \mathbf{r_2}(s), ..., \mathbf{r_n}(s)]$. The columns of $\mathbf{S}(s)\mathbf{W_{p1}}(s)$ are the errors from the n experiments. For step inputs, one can take $\mathbf{W_{p1}}(s) = s^{-1}\mathbf{I}(s)$.

$\mathbf{W_{p2}}(s)$ is the output weighting function. Premultiplication by the output weight $\mathbf{W_{p2}}(s)$ generates $\mathbf{W_{p2}}(s)\mathbf{S}(s)\mathbf{W_{p1}}(s)$. The columns of the matrix are the weighted errors from the n experiments. The output weight is used since it may be desirable to make errors small over some frequency ranges.

In the method of this book, a filter is introduced to achieve the same goal. Hence, $\mathbf{W_{p2}}(s) = \mathbf{I}$ is taken. Comparatively, it is simple to penalize errors by a filter; the resulting system is easy to tune.

Now only one weighting function needs to be considered. Hence, $\mathbf{W_{p1}}(s)$ is denoted by $\mathbf{W}(s)$ for simplicity

10.5 Robust Stability of MIMO Systems

The description of uncertainty and the test of robustness for MIMO plants are very involved. The result is not a simple generalization of the SISO case

SISO case: The uncertainty is described by an uncertain plant family. The family corresponds to a Nyquist band consisting of a union of disks with specified radius
MIMO case: Similar uncertainty description can be developed for MIMO systems. Since the commutative property does not hold for matrix multiplication, one has to distinguish the uncertainty occurring at the plant input and that at the plant output

Assume that the uncertain plants have the same number of RHP poles as the nominal plant. Let the subscript "I" stand for "Input" and the subscript "O" stand for "Output"

The uncertain plant can be described in the following manner:

Plant with multiplicative output uncertainty:

$$\tilde{\mathbf{G}}(s) = [\mathbf{I} + \delta_{\mathbf{O}}(s)]\mathbf{G}(s)$$

Plant with multiplicative input uncertainty:

$$\tilde{\mathbf{G}}(s) = \mathbf{G}(s)[\mathbf{I} + \delta_{\mathbf{I}}(s)]$$

Both of the two uncertainties can be described in a unified form:

$$\delta(s) = \mathbf{W}_2(s)\boldsymbol{\Delta}(s)\mathbf{W}_1(s)$$

where $\mathbf{W}_1(s)$ and $\mathbf{W}_2(s)$ are stable weighting function matrices. $\boldsymbol{\Delta}(s)$ is a stable transfer function matrix denoting the normalized uncertainty:

$$\bar{\sigma}[\boldsymbol{\Delta}(j\omega)] \leq 1, \quad \forall\omega$$

or equivalently,

$$\|\boldsymbol{\Delta}(s)\|_\infty \leq 1$$

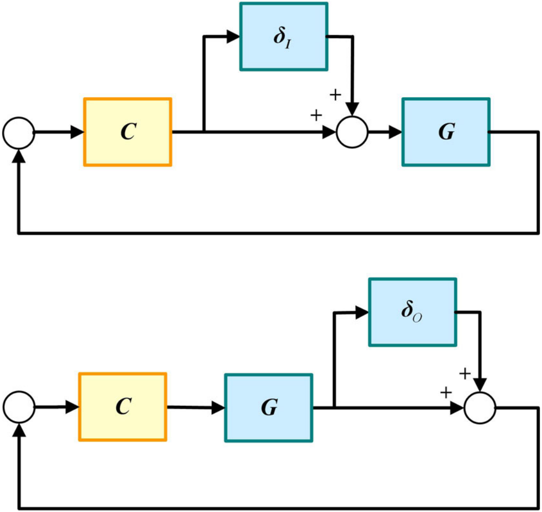


Figure: Input uncertainty $\delta_I(s)$ and output uncertainty $\delta_O(s)$

Unstructured uncertainty: Constructed by lumping different sources of uncertainties into a single uncertainty
Let $\Delta_m(s)$ be a stable scalar weighting function. The unstructured uncertainty is usually interpreted as follows:

$$\delta(s) = \Delta_m(s)\boldsymbol{\Delta}(s)$$

that is, in the unified form of uncertainty

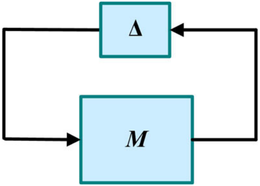
$$\mathbf{W_1}(s)(\text{or } \mathbf{W_2}(\mathrm{s})) = \Delta_\mathrm{m}(\mathrm{s}), \mathbf{W_2}(\mathrm{s})(\text{or } \mathbf{W_1}(\mathrm{s})) = 1$$

$\Delta_m(s)$ gives the profile for the magnitude of the uncertainty $\delta(s)$:

$$\bar{\sigma}\left[\delta(j\omega)\right] \leq |\Delta_m(j\omega)|, \quad \forall\omega$$

$\tilde{\mathbf{G}}(j\omega)$ describes a disk with the center $\mathbf{G}(j\omega)$ and the radius $|\Delta_m(j\omega)|$ at each frequency ω

To analyze the robust stability of the closed-loop system in a unified framework, the system is usually redrawn in the $\mathbf{M\Delta}$ form



Let

$$\begin{aligned} \mathbf{S_I}(s) &= [\mathbf{I} + \mathbf{C}(s)\mathbf{G}(s)]^{-1} \\ \mathbf{T_I}(s) &= [\mathbf{I} + \mathbf{C}(s)\mathbf{G}(s)]^{-1}\mathbf{C}(s)\mathbf{G}(s) \end{aligned}$$

and

$$\begin{aligned} \mathbf{S_O}(s) &= [\mathbf{I} + \mathbf{G}(s)\mathbf{C}(s)]^{-1} \\ \mathbf{T_O}(s) &= \mathbf{G}(s)\mathbf{C}(s)[\mathbf{I} + \mathbf{G}(s)\mathbf{C}(s)]^{-1} \end{aligned}$$

It is easy to verify that for the input uncertainty $\delta_I(s) = \Delta_{Im}(s)\mathbf{\Delta}_I(s)$,

$$\mathbf{M}(s) = -\mathbf{T_I}(s)\Delta_{Im}(s), \mathbf{\Delta}(s) = \mathbf{\Delta}_I(s)$$

and for the output uncertainty $\delta_O(s) = \Delta_{Om}(s)\mathbf{\Delta}_O(s)$,

$$\mathbf{M}(s) = -\mathbf{T_O}(s)\Delta_{Om}(s), \mathbf{\Delta}(s) = \mathbf{\Delta}_O(s)$$

When the nominal system is internally stable, $\mathbf{M}(s)$ is stable. The following theorem gives the condition for the robust stability

Theorem

The closed-loop system shown is stable for all $\mathbf{\Delta}(s)$s if and only if one of the following two equivalent conditions is satisfied:

1. $\det[\mathbf{I} - \mathbf{M}(j\omega)\mathbf{\Delta}(j\omega)] \neq 0, \quad \forall\omega, \forall\mathbf{\Delta}(j\omega)$

2. $\|\mathbf{M}(s)\|_\infty < 1$

Proof.

1. By assumption, the nominal system is internally stable; the uncertain plant and the nominal plant have the same number of RHP poles. The closed-loop system shown in Figure **??** is stable for all $\mathbf{\Delta}(s)$s, if and only if $\det[\mathbf{I} + \mathbf{M}(s)\mathbf{\Delta}(s)]$ encircle the origin as many times as the nominal system.
If the Nyquist plot of $\det[\mathbf{I} + \mathbf{M}(s)\mathbf{\Delta}(s)]$ does not pass through the origin, the number of encirclements will not change. This is equivalent to

$$\det[\mathbf{I} - \mathbf{M}(j\omega)\mathbf{\Delta}(j\omega)] \neq 0, \quad \forall\omega, \forall\mathbf{\Delta}(j\omega)$$

2. The result can be proved by contradiction.
First, it is shown that $\rho[\mathbf{M}(j\omega)\mathbf{\Delta}(j\omega)] < 1$ is sufficient. Assume that there exist a frequency ω' and an uncertainty $\mathbf{\Delta}'(j\omega')$ such that $\rho[\mathbf{M}(j\omega')\mathbf{\Delta}'(j\omega')] < 1$, but

Proof ctd.1.

$$\det[\mathbf{I} - \mathbf{M}(j\omega')\epsilon\boldsymbol{\Delta}'(j\omega')] = 0$$

which is equivalent to

$$\prod_i \lambda_{ei}[\mathbf{I} - \mathbf{M}(j\omega')\boldsymbol{\Delta}'(j\omega')] = 0$$

This implies that for some i

$$1 - \lambda_{ei}[\mathbf{M}(j\omega')\boldsymbol{\Delta}'(j\omega')] = 0$$

Then

$$\rho[\mathbf{M}(j\omega')\boldsymbol{\Delta}'(j\omega')] \geq 1$$

Proof ctd.2.

which is a contradiction. Therefore, $\rho[\mathbf{M}(j\omega)\mathbf{\Delta}(j\omega)] < 1$ is sufficient for robust stability.
Because $\rho[\mathbf{M}(j\omega)\mathbf{\Delta}'(j\omega)] \leq \|\mathbf{M}(j\omega)\mathbf{\Delta}'(j\omega)\|_\infty \leq \|\mathbf{M}(s)\|_\infty$, $\|\mathbf{M}(s)\|_\infty < 1$ is also sufficient.
To prove the necessary of $\|\mathbf{M}(s)\|_\infty < 1$, assume that the closed-loop system is stable, but $\|\mathbf{M}(s)\|_\infty \geq 1$. For some frequency ω' we have $\sigma_1[\mathbf{M}(j\omega')] \geq 1$. It will be shown that there exists a $\mathbf{\Delta}'(s)$ with $\left\|\mathbf{\Delta}'(s)\right\|_\infty \leq 1$ such that the closed-loop system is unstable.
Let the SVD of $\mathbf{M}(j\omega')$ be

$$\mathbf{M}(j\omega') = \mathbf{U}(j\omega')\mathbf{\Sigma}(j\omega')\mathbf{V}^H(j\omega')$$

Proof ctd.3.

Define

$$\mathbf{D}(j\omega') = \mathrm{diag}\{1/\sigma_1[\mathbf{M}(j\omega')], 0, ..., 0\}$$

and

$$\mathbf{\Delta}'(s) = \mathbf{V}(s)\mathbf{D}(j\omega')\mathbf{U}^H(s)$$

$\mathbf{V}(s)$ and $\mathbf{U}(s)$ can easily be constructed from the complex matrices $\mathbf{V}(j\omega')$ and $\mathbf{U}(j\omega')$. The first vector of $\mathbf{V}(j\omega')$, $\mathbf{v}_1(j\omega')$, is used to illustrate the procedure. Write it in the following form:

$$\mathbf{v}_1^T(j\omega') = [v_{11}e^{j\phi_1}, v_{12}e^{j\phi_2}, ..., v_{1n}e^{j\phi_n}]$$

where v_{1j} are real numbers, and are chosen so that $\phi_j \in [-\pi, 0)$, $j = 1, 2, ..., n$. Choose $\alpha_j \geq 0$ so that

Proof ctd.4.

$$\angle\left(\frac{\alpha_j - j\omega'}{\alpha_j + j\omega'}\right) = \phi_j$$

$\mathbf{v}_1(s)$ can be taken as

$$\mathbf{v}_1^T(s) = \left[v_{11}\frac{\alpha_1 - s}{\alpha_1 + s}, v_{12}\frac{\alpha_2 - s}{\alpha_2 + s}, \ldots, v_{1n}\frac{\alpha_n - s}{\alpha_n + s}\right]$$

Clearly,

$$\left\|\mathbf{\Delta}'(s)\right\|_\infty = 1/\sigma_1[\mathbf{M}(j\omega')] \leq 1$$

and

Proof ctd.5.

$$\begin{aligned} & \det[\mathbf{I} - \mathbf{M}(j\omega')\mathbf{\Delta}'(j\omega')] \\ = \quad & \det[\mathbf{I} - \mathbf{U}(j\omega')\mathbf{\Sigma}(j\omega')\mathbf{V}^H(j\omega')\mathbf{V}(j\omega')\mathbf{D}(j\omega')\mathbf{U}^H(j\omega')] \\ = \quad & \det[\mathbf{I} - \mathbf{U}(j\omega')\mathbf{\Sigma}(j\omega')\mathbf{D}(j\omega')\mathbf{U}^H(j\omega')] \\ = \quad & 0, \end{aligned}$$

which implies the closed-loop system is unstable □

In particular, for input uncertainty and output uncertainty, there are the following results.

Corollary

The closed-loop system is stable for the multiplicative input uncertainty if and only if

$$\|\mathbf{T_I}(s)\Delta_{Im}(s)\|_\infty < 1$$

Corollary

The closed-loop system is stable for the multiplicative output uncertainty if and only if

$$\|\mathbf{T_O}(s)\Delta_{Om}(s)\|_\infty < 1$$

Structured uncertainty: Sometimes, the unstructured uncertainty description is conservative. In this case, it is desirable to use the structured uncertainty description. Unfortunately, in most cases simple and meaningful conditions cannot be obtained for a rigorous structured uncertainty description. A compromise is that some sources of uncertainties are described in a structured manner, while the rest is lumped into a single unstructured uncertainty

Corollary

The closed-loop system is stable for the multiplicative output uncertainty if and only if

$$\left\| \mathbf{T_O}(s)\Delta_{Om}(s) \right\|_\infty < 1$$

Structured uncertainty: Sometimes, the unstructured uncertainty description is conservative. In this case, it is desirable to use the structured uncertainty description. Unfortunately, in most cases simple and meaningful conditions cannot be obtained for a rigorous structured uncertainty description. A compromise is that some sources of uncertainties are described in a structured manner, while the rest is lumped into a single unstructured uncertainty

The combined unstructured/structured uncertainty is usually expressed in the form of a large block diagonal matrix:

$$\delta(s) = \mathbf{W_2}(s)\boldsymbol{\Delta}(s)\mathbf{W_1}(s)$$

with

$$\begin{aligned}
\boldsymbol{\Delta}(s) &= \mathrm{diag}\{\boldsymbol{\Delta}_\mathbf{1}(s), \boldsymbol{\Delta}_\mathbf{2}(s), ..., \boldsymbol{\Delta}_\mathbf{m}(s)\} \\
\|\boldsymbol{\Delta}_i(s)\|_\infty &\leq 1, i = 1, 2, ..., m \\
\mathbf{W_1}(s) &= \mathrm{diag}\{\mathbf{W_{11}}(s), \mathbf{W_{12}}(s), ..., \mathbf{W_{1m}}(s)\} \\
\mathbf{W_2}(s) &= \mathrm{diag}\{\mathbf{W_{21}}(s), \mathbf{W_{22}}(s), ..., \mathbf{W_{2m}}(s)\}
\end{aligned}$$

$\mathbf{W_1}(s)$ and $\mathbf{W_2}(s)$ are stable transfer function matrices

It can be proved that the robust stability is guaranteed if and only if

$$\det[\mathbf{I} - \mathbf{M}(j\omega)\boldsymbol{\Delta}(j\omega)] \neq 0, \quad \forall\omega, \forall\boldsymbol{\Delta}(j\omega)$$

Nevertheless, the second condition, namely $\|\mathbf{M}(s)\|_\infty < 1$, is only **sufficient** for robust stability, since only those having the specific block diagonal structure are permissible

This can be conservative. To deal with this problem, Structured Singular Value (SSV) is proposed. It can be regarded as a generalization of the singular value

Definition

Find the smallest $\bar{\sigma}[\mathbf{\Delta}(j\omega)](\|\mathbf{\Delta}(s)\|_\infty \leq k_m)$ that makes $\mathbf{I} - \mathbf{M}(j\omega)\mathbf{\Delta}(j\omega)$ singular. The SSV $\mu[\mathbf{M}(j\omega)] = 1/\bar{\sigma}[\mathbf{\Delta}(j\omega)]$, or,

$$\mu^{-1}[\mathbf{M}(j\omega)] = \min\{\bar{\sigma}[\mathbf{\Delta}(j\omega)] : \det[\mathbf{I} - \mathbf{M}(j\omega)\mathbf{\Delta}(j\omega)] = 0, \forall \mathbf{\Delta}(j\omega)\}$$

If no $\mathbf{\Delta}(j\omega)$ exists such that $\det[\mathbf{I} - \mathbf{M}(j\omega)\mathbf{\Delta}(j\omega)] = 0$, then $\mu[\mathbf{M}(j\omega)] = 0$

The singularity of a complex matrix means that its determinant is zero

It is noted that the SSV depends on not only $\mathbf{M}(j\omega)$ but also the structure of $\mathbf{\Delta}(j\omega)$

At present, the SSV can only be computed numerically

In the case where $\mathbf{\Delta}(j\omega)$ is unstructured (that is, it is a full matrix), $\mu[\mathbf{M}(j\omega)] = \bar{\sigma}[\mathbf{M}(j\omega)]$

Theorem

The closed-loop system is stable for all $\mathbf{\Delta}(s)s(\|\mathbf{\Delta}(s)\|_\infty \leq 1)$ if and only if

$$\mu[\mathbf{M}(j\omega)] < 1, \quad \forall\omega$$

Proof.

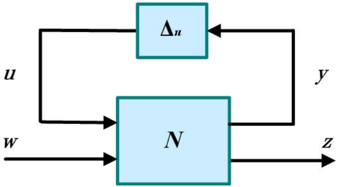
If $\mu[\mathbf{M}(j\omega)] < 1$ at all frequencies, then $\bar{\sigma}[\mathbf{\Delta}(j\omega)] > 1$, which implies that no permissible $\mathbf{\Delta}(j\omega)$ exists such that $\det[\mathbf{I} - \mathbf{M}(j\omega)\mathbf{\Delta}(j\omega)] = 0$. Hence, the system is stable for all $\mathbf{\Delta}(s)s(\|\mathbf{\Delta}(s)\|_{\infty} \leq 1)$.

Assume that the system is stable, but $\mu[\mathbf{M}(j\omega')] \geq 1$ at some frequency ω'. From the definition of the SSV, there must exist an uncertainty $\bar{\sigma}[\mathbf{\Delta}(j\omega')] \leq 1$ so that $\det[\mathbf{I} - \mathbf{M}(j\omega')\mathbf{\Delta}(\omega')] = 0$. Then the system is unstable. This contradicts with the assumption

□

10.6 Robust Performance of MIMO Systems

In some design methods for MIMO control systems, the general control configuration is frequently used. The block $\mathbf{N}(s)$ in the configuration has two sets of inputs and two sets of outputs



The first set of inputs— All exogenous signals (such as references or disturbances)

The first set of outputs—The outputs whose behavior is of interest (such as plant outputs or error signals)

The second set of inputs and outputs—The outputs and inputs of the uncertainty of the plant, respectively

Assume that the uncertainty is expressed as a block diagonal matrix:

$$\mathbf{\Delta_u}(s) = \mathrm{diag}\{\mathbf{\Delta_1}(s), \mathbf{\Delta_2}(s), ..., \mathbf{\Delta_m}(s)\}$$

with

$$\|\mathbf{\Delta_u}(s)\|_\infty \leq 1$$

The subscript "u" stands for "uncertainty". $\mathbf{N}(s)$ is stable when the nominal system is internally stable. Partition $\mathbf{N}(s)$ as

$$\mathbf{N}(s) = \begin{bmatrix} \mathbf{N}_{11}(s) & \mathbf{N}_{12}(s) \\ \mathbf{N}_{21}(s) & \mathbf{N}_{22}(s) \end{bmatrix}$$

with the dimensions of its parts compatible with the input and output signals. The transfer function matrix from $\mathbf{w}(s)$ to $\mathbf{z}(s)$, $\mathbf{F}(\mathbf{N}, \mathbf{\Delta_u})$, can be expressed in the form of Linear Fractional Transformation (LFT):

$$\mathbf{F}(\mathbf{N}, \mathbf{\Delta_u}) = \mathbf{N}_{22}(s) + \mathbf{N}_{21}(s)\mathbf{\Delta}(s)[\mathbf{I} - \mathbf{N}_{11}(s)\mathbf{\Delta}(s)]^{-1}\mathbf{N}_{12}(s)$$

This can be obtained by eliminating $\mathbf{y}(s)$ and $\mathbf{u}(s)$ from the following equations:

$$\begin{aligned} \begin{bmatrix} \mathbf{y}(s) \\ \mathbf{z}(s) \end{bmatrix} &= \begin{bmatrix} \mathbf{N}_{11}(s) & \mathbf{N}_{12}(s) \\ \mathbf{N}_{21}(s) & \mathbf{N}_{22}(s) \end{bmatrix} \begin{bmatrix} \mathbf{u}(s) \\ \mathbf{w}(s) \end{bmatrix} \\ \mathbf{u}(s) &= \mathbf{\Delta}(s)\mathbf{y}(s) \end{aligned}$$

In particular, when there is no uncertainty,

$$\mathbf{F}(\mathbf{N}, \mathbf{\Delta_u}) = \mathbf{N}_{22}(s)$$

i.e. the nominal transfer function matrix from $\mathbf{w}(s)$ to $\mathbf{z}(s)$ is $\mathbf{N}_{22}(s)$

In $\mathbf{F}(\mathbf{N}, \mathbf{\Delta_u})$, the only possible source of instability is the term $[\mathbf{I} - \mathbf{N}_{11}(s)\mathbf{\Delta}(s)]^{-1}$ when the nominal system is stable. Identifying $\mathbf{N}_{11}(s)$ with $\mathbf{M}(s)$, the stability of the system can be tested by utilizing the $\mathbf{M} - \mathbf{\Delta}$ strucutre

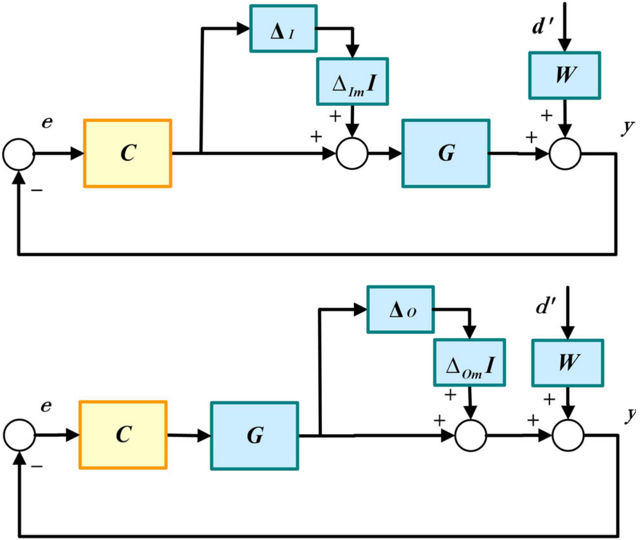


Figure: Systems with input or output uncertainty

Consider the uncertain systems shown in Figure, where $\mathbf{W}(s)$ is the input weighting function. Define $\mathbf{z}(s) = \mathbf{y}(s)$ and $\mathbf{w}(s) = \mathbf{d}'(s)$. $\mathbf{F}(\mathbf{N}, \boldsymbol{\Delta}_\mathbf{u})$ is the perturbed weighting sensitivity function. For the input uncertainty, it is not difficult to convert the diagram into the $\mathbf{N} - \boldsymbol{\Delta}$ strucutre:

$$\mathbf{N}(s) = \begin{bmatrix} \mathbf{C}(s)\mathbf{S}_\mathbf{I}(s)\mathbf{G}(s)\Delta_{Im}(s) & -\mathbf{C}(s)\mathbf{S}_\mathbf{I}(s)\mathbf{W}(s) \\ \mathbf{S}_\mathbf{I}(s)\mathbf{G}(s)\Delta_{Im}(s) & \mathbf{S}_\mathbf{I}(s)\mathbf{W}(s) \end{bmatrix}$$

With respect to the output uncertainty, we have

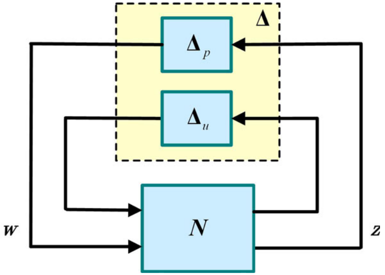
$$\mathbf{N}(s) = \begin{bmatrix} \mathbf{T}_\mathbf{O}(s)\Delta_{Om}(s) & -\mathbf{T}_\mathbf{O}(s)\mathbf{W}(s) \\ \mathbf{S}_\mathbf{O}(s)\Delta_{Om}(s) & \mathbf{S}_\mathbf{O}(s)\mathbf{W}(s) \end{bmatrix}$$

Robust performance means that the performance objective is satisfied even there exists uncertainty. Suppose the robust performance is measured in terms of the ∞ norm, like in the SISO case

The robust performance can be expressed as

$$\|\mathbf{F}(\mathbf{N}, \boldsymbol{\Delta}_\mathbf{u})\|_\infty < 1$$

To analyze the robust performance, an uncertainty $\boldsymbol{\Delta}_\mathbf{p}(s)$ $(\bar{\sigma}[\boldsymbol{\Delta}_\mathbf{p}(j\omega)] \leq 1)$ is introduced, as shown in Figure. Here the subscript p stands for "performance". $\boldsymbol{\Delta}_\mathbf{p}(s)$ is a fictitious uncertainty



The reason of introducing such an uncertainty is to build a relationship between the **NΔ** structure and the **MΔ** structure. In this way, the preceding result can be utilized to derive the condition for testing robust performance

Now the new uncertainty can be written as

$$\begin{aligned}\boldsymbol{\Delta}(s) &= \mathrm{diag}\{\boldsymbol{\Delta}_{\mathbf{u}}(s), \boldsymbol{\Delta}_{\mathbf{p}}(s)\} \\ \|\boldsymbol{\Delta}(s)\|_\infty &\leq 1\end{aligned}$$

The following theorem can readily be obtained.

Theorem

The stable system $\mathbf{N}(s)$ *satisfies the robust performance condition* $\|\mathbf{F}(\mathbf{N}, \boldsymbol{\Delta}_{\mathbf{u}})\|_\infty < 1$ *if and only if*

$$\mu_\Delta[\mathbf{N}(j\omega)] < 1, \forall\omega,$$

where μ *is computed with respect to the block diagonal uncertainty* $\boldsymbol{\Delta}(s)$

End of Chapter 10