Chapter 13 H_2 Decoupling Control

13.1 Controller Parameterization for MIMO Systems

H_∞ control and H_2 control are two prevailing design methods

Merit of H_∞ control: In Section 3.4, it was seen that the sufficient and necessary condition for robust performance analysis could be obtained when the system performance is specified in terms of the ∞-norm

Merit of H_2 control: For controller design, however, the 2-norm is mathematically more convenient for treatment

Methodology of SISO quasi-H_∞ control: The controller is designed by constructing a desired closed-loop transfer function

Methodology of SISO H_2 control: The controller is derived by solving an optimization problem

In this chapter, the SISO H_2 design will be extended to the decoupling control:

1. All stabilizing decoupling controllers are parameterized
2. Diagonal factorizations are defined for MIMO plants
3. Based on the parameterization and the factorization, the controller and the decoupler are analytically derived in one step

Consider the control system consisting of an $n \times n$ plant $\mathbf{G}(s)$ and an $n \times n$ controller $\mathbf{C}(s)$.

$$\mathbf{C}(s) = \mathbf{Q}(s)[\mathbf{I} - \mathbf{G}(s)\mathbf{Q}(s)]^{-1}$$

where $\mathbf{Q}(s)$ is the IMC controller

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In the design method of H_2 decoupling control in this chapter, the plant can be proper, have time delays, have poles on the imaginary axis, or have poles and zeros in the open RHP

It is assumed that

1. There is not any unstable hidden mode in $\mathbf{G}(s)$.
2. $\mathbf{G}(s)$ is of full normal rank.
3. $\mathbf{G}(s)$ does not have any finite zeros on the imaginary axis.

The first two assumptions are the same as those in the quasi-H_∞ decoupling control. The third assumption is necessary for H_2 decoupling control, because zeros on the imaginary axis may cause internal instability

Practical plants seldom have finite zeros on the imaginary axis. In case this happens, to use the design method here a slight perturbation can be introduced to the zeros. For example, substitute $(s+0.01)/(s+1)$ for $s/(s+1)$

As we known, the closed-loop system is internally stable iff

1. $\mathbf{Q}(s)$ is stable
2. $[\mathbf{I}-\mathbf{G}(s)\mathbf{Q}(s)]\mathbf{G}(s)$ is stable

Theorem

Assume that $\mathbf{G}(s)$ is a plant with time delays. The unity feedback control system is internally stable if and only if

1. $\mathbf{Q}(s)$ *is stable*
2. $\mathbf{I}-\mathbf{G}(s)\mathbf{Q}(s)$ *has zeros wherever* $\mathbf{G}(s)$ *has unstable poles*
3. *All the RHP zero-pole cancellations in* $[I-\mathbf{G}(s)\mathbf{Q}(s)]\mathbf{G}(s)$ *are removed*

Proof.

It is enough to prove the equivalence between the stability of $[\mathbf{I} - \mathbf{G}(s)\mathbf{Q}(s)]\mathbf{G}(s)$ and the second and third conditions

Assume that $[\mathbf{I} - \mathbf{G}(s)\mathbf{Q}(s)]\mathbf{G}(s)$ is stable. Evidently, $\mathbf{I} - \mathbf{G}(s)\mathbf{Q}(s)$ must have zeros wherever $\mathbf{G}(s)$ has unstable poles and all the RHP zero-pole cancellations in $[I - \mathbf{G}(s)\mathbf{Q}(s)]\mathbf{G}(s)$ are removed. Otherwise, $[\mathbf{I} - \mathbf{G}(s)\mathbf{Q}(s)]\mathbf{G}(s)$ will have unstable poles, which contradicts the assumption

Now, assume that $\mathbf{I} - \mathbf{G}(s)\mathbf{Q}(s)$ has zeros wherever $\mathbf{G}(s)$ has unstable poles and all the RHP zero-pole cancellations in $[I - \mathbf{G}(s)\mathbf{Q}(s)]\mathbf{G}(s)$ are removed. As the only possible unstable poles of $[\mathbf{I} - \mathbf{G}(s)\mathbf{Q}(s)]\mathbf{G}(s)$ are those of $\mathbf{G}(s)$, the second assumption implies that all the unstable poles in $[\mathbf{I} - \mathbf{G}(s)\mathbf{Q}(s)]\mathbf{G}(s)$ are removed by the zeros of $\mathbf{I} - \mathbf{G}(s)\mathbf{Q}(s)$ □

In multivariable systems, there exists such a possibility that $\mathbf{I} - \mathbf{G}(s)\mathbf{Q}(s)$ has zeros wherever $\mathbf{G}(s)$ has unstable poles, but there are irremovable RHP zero-pole cancellations in $[\mathbf{I} - \mathbf{G}(s)\mathbf{Q}(s)]\mathbf{G}(s)$. **Only when there is not any irremovable RHP zero-pole cancellation in $[\mathbf{I} - \mathbf{G}(s)\mathbf{Q}(s)]\mathbf{G}(s)$**, can the stability of $[\mathbf{I} - \mathbf{G}(s)\mathbf{Q}(s)]\mathbf{G}(s)$ be guaranteed

Example

This example illustrates the irremovable RHP zero-pole cancellation in $[\mathbf{I} - \mathbf{G}(s)\mathbf{Q}(s)]\mathbf{G}(s)$. The plant is described by the transfer function matrix

$$\mathbf{G}(s) = \begin{bmatrix} \frac{1}{s+3} & \frac{1}{s-2} \\ \frac{2}{s+3} & \frac{s-1}{s-2} \end{bmatrix}$$

It has two poles at $s = -3$ and $s = 2$, and one zero at $s = 3$

Example (ctd.1)

Assume that the controller is

$$\mathbf{Q}(s) = \begin{bmatrix} \frac{-(s-1)}{s+1} & \frac{22s+1}{(s+1)^2} \\ \frac{2(s-2)}{(s+3)(s+1)} & \frac{-(s-2)(22s+1)}{(s+3)(s+1)^2} \end{bmatrix}$$

$\mathbf{Q}(s)$ is stable. Since

$$\mathbf{I} - \mathbf{G}(s)\mathbf{Q}(s) = \begin{bmatrix} \frac{s(s+5)}{(s+3)(s+1)} & 0 \\ 0 & \frac{s(s+29)(s-2)}{(s+3)(s+1)^2} \end{bmatrix}$$

$\mathbf{I} - \mathbf{G}(s)\mathbf{Q}(s)$ has zeros wherever $\mathbf{G}(s)$ has unstable poles

Example (ctd.2)

Nevertheless,

$$[\mathbf{I} - \mathbf{G}(s)\mathbf{Q}(s)]\mathbf{G}(s) = \begin{bmatrix} \frac{s(s+5)}{(s+3)^2(s+1)} & \frac{s(s+5)}{(s-2)(s+3)(s+1)} \\ \frac{2s(s+29)(s-2)}{(s+3)^2(s+1)^2} & \frac{s(s+29)(s-1)}{(s+3)(s+1)^2} \end{bmatrix}$$

is not stable. It has a RHP pole at $s = 2$ and a RHP zero at $s = 2$. The RHP zero-pole cancellation cannot be removed

Let the multiplicity of the unstable pole p_j ($\mathrm{Re}(p_j) \geq 0, j = 1, 2, ..., r_p$) be l_j, and let l_{ij} be the largest multiplicity of p_j in the ith row of $\mathbf{G}(s)$

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Theorem

All controllers that make the unity feedback control system internally stable can be parameterized as

$$\mathbf{C}(s) = \mathbf{Q}(s)[\mathbf{I} - \mathbf{G}(s)\mathbf{Q}(s)]^{-1}$$

where $\mathbf{Q}(s)$ *is any stable proper transfer function matrix that satisfies*

$$\lim_{s \to p_j} \frac{d^k}{ds^k} \det[\mathbf{I} - \mathbf{G}(s)\mathbf{Q}(s)] = 0, j = 1, 2, ..., r_p; k = 0, 1, ..., l_j - 1$$

and all the RHP zero-pole cancellations in $[I - \mathbf{G}(s)\mathbf{Q}(s)]\mathbf{G}(s)$ *are removed*

Proof.

To guarantee the internal stability of the closed-loop system, first, $\mathbf{Q}(s)$ should be stable. This implies that $\mathbf{Q}(s)$ should be proper

Second, $[\mathbf{I}-\mathbf{G}(s)\mathbf{Q}(s)]\mathbf{G}(s)$ should be stable. This implies that $\mathbf{I}-\mathbf{G}(s)\mathbf{Q}(s)$ has to cancel all the closed RHP poles of $\mathbf{G}(s)$. To achieve this $\mathbf{Q}(s)$ must satisfy that

$$\lim_{s\to p_j}\frac{d^k}{ds^k}\det[\mathbf{I}-\mathbf{G}(s)\mathbf{Q}(s)]=0, j=1,2,...,r_p; k=0,1,...,l_j-1$$

The condition cannot guarantee the stability of $[\mathbf{I}-\mathbf{G}(s)\mathbf{Q}(s)]\mathbf{G}(s)$ unless all the RHP zero-pole cancellations in $[I-\mathbf{G}(s)\mathbf{Q}(s)]\mathbf{G}(s)$ are removed $\square$

Corollary

Assume that $\mathbf{G}(s)$ *is a stable plant. All controllers that make the unity feedback control system internally stable can be parameterized as*

$$\mathbf{C}(s) = \mathbf{Q}(s)[\mathbf{I} - \mathbf{G}(s)\mathbf{Q}(s)]^{-1}$$

where $\mathbf{Q}(s)$ *is any stable proper transfer function matrix*

It is noted that for stable plants the new paramerization is identical to the Youla parameterization

When the system performance is considered, it is always desirable that the system possesses the asymptotic tracking property

Corollary

Assume that $\mathbf{G}(s)$ *is a stable plant. All controllers that make the unity feedback control system internally stable can be parameterized as*

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When the system performance is considered, it is always desirable that the system possesses the asymptotic tracking property

If the system inputs are steps, the closed-loop transfer function matrix $\mathbf{T}(s) = \mathbf{G}(s)\mathbf{Q}(s)$ should satisfy the following condition for the asymptotic tracking property:

$$\lim_{s \to 0} [\mathbf{I} - \mathbf{T}(s)] = \mathbf{0}$$

Theorem

All controllers that make the unity feedback control system internally stable and possess the asymptotic tracking property for step inputs can be parameterized as

$$\mathbf{C}(s) = \mathbf{Q}(s)[\mathbf{I} - \mathbf{G}(s)\mathbf{Q}(s)]^{-1}$$

where

$$\mathbf{Q}(s) = \mathbf{G}^{-1}(0)[\mathbf{I} + s\mathbf{Q_1}(s)]$$

Theorem (ctd.1)

$\mathbf{Q_1}(s)$ *is any stable transfer function matrix that makes* $\mathbf{Q}(s)$ *proper and satisfies*

$$\lim_{s\to p_j} \frac{d^k}{ds^k} \det[\mathbf{I} - \mathbf{G}(s)\mathbf{G}^{-1}(0) - s\mathbf{G}(s)\mathbf{G}^{-1}(0)\mathbf{Q_1}(s)] = 0,$$

$$j = 1, 2, ..., r_p; k = 0, 1, ..., l_j - 1$$

and all the RHP zero-pole cancellations in $[I - \mathbf{G}(s)\mathbf{Q}(s)]\mathbf{G}(s)$ *are removed*

Proof.

If

$$\lim_{s\to 0}[\mathbf{I} - \mathbf{G}(s)\mathbf{Q}(s)] = \mathbf{0}$$

Proof ctd.1.

or equivalently,

$$\mathbf{Q}(0) = \mathbf{G}^{-1}(0)$$

the closed-loop system possesses the asymptotic tracking property. All transfer function matrices that satisfy the condition can be written as

$$\mathbf{Q}(s) = \mathbf{G}^{-1}(0)[\mathbf{I} + s\mathbf{Q_1}(s)]$$

To guarantee the internal stability of the closed-loop system, $\mathbf{Q}(s)$ should be stable. This implies that $\mathbf{Q}(s)$ is proper. Second, $\mathbf{I} - \mathbf{G}(s)\mathbf{Q}(s)$ has to cancel all the closed RHP poles of $\mathbf{G}(s)$. To achieve this, $\mathbf{Q}(s)$ must satisfy that

Proof ctd.2.

$$\lim_{s \to p_j} \frac{d^k}{ds^k} \det[\mathbf{I} - \mathbf{G}(s)\mathbf{G}^{-1}(0) - s\mathbf{G}(s)\mathbf{G}^{-1}(0)\mathbf{Q_1}(s)] = 0$$

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The condition cannot guarantee the stability of $[\mathbf{I} - \mathbf{G}(s)\mathbf{Q}(s)]\mathbf{G}(s)$ unless all the RHP zero-pole cancellations in $[I - \mathbf{G}(s)\mathbf{Q}(s)]\mathbf{G}(s)$ are removed □

It is easy to obtain the closed-loop transfer function matrix:

$$\mathbf{T}(s) = \mathbf{G}(s)\mathbf{Q}(s) = \mathbf{G}(s)\mathbf{G}^{-1}(0)[\mathbf{I} + s\mathbf{Q_1}(s)]$$

In practice, it may be required that the closed-loop response is decoupled. Let the ith element of $\mathbf{T}(s)$ be $T_i(s)$

Proof ctd.2.

$$\lim_{s \to p_j} \frac{d^k}{ds^k} \det[\mathbf{I} - \mathbf{G}(s)\mathbf{G}^{-1}(0) - s\mathbf{G}(s)\mathbf{G}^{-1}(0)\mathbf{Q_1}(s)] = 0$$

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In practice, it may be required that the closed-loop response is decoupled. Let the ith element of $\mathbf{T}(s)$ be $T_i(s)$

Corollary

Assume that the closed-loop response is decoupled. All controllers that make the unity feedback control system internally stable and possess the asymptotic tracking property for step inputs can be parameterized as

$$\mathbf{C}(s) = \mathbf{Q}(s)[\mathbf{I} - \mathbf{G}(s)\mathbf{Q}(s)]^{-1}$$

where

$$\mathbf{Q}(s) = \mathbf{G}^{-1}(0)[\mathbf{I} + s\mathbf{Q_1}(s)]$$

$\mathbf{Q_1}(s)$ *is any stable transfer function matrix that makes* $\mathbf{Q}(s)$ *proper and* $\mathbf{T}(s)$ *diagonal, and satisfies*

Corollary (ctd.1)

$$\lim_{s \to p_j} \frac{d^k}{ds^k}[1 - T_i(s)] = 0$$

$$i = 1, 2, ..., n; j = 1, 2, ..., r_p; k = 0, 1, ..., l_{ij} - 1$$

and all the RHP zero-pole cancellations in $[I - \mathbf{G}(s)\mathbf{Q}(s)]\mathbf{G}(s)$ *are removed*

The well-known Youla parameterization is not used here because of several reasons:

1. It cannot be directly used for plants with time delays
2. It needs a coprime factorization, which cannot be obtained by means of analytical methods
3. It does not directly relate to the IMC controller $\mathbf{Q}(s)$
4. It cannot be directly used for decoupling control

13.2 Diagonal Factorization for H_2 Control

The diagonal factorization for H_2 decoupling control: The factorization is similar to, but not exactly the same as that for quasi-H_∞ decoupling control

The introduction follows the original developing procedure of the factorization, which explains why such a factorization is constructed

Assume that the plant is expressed as

$$\mathbf{G}(s) = \begin{bmatrix} G_{11}(s)e^{-\theta_{11}s} & \cdots & G_{1n}(s)e^{-\theta_{1n}s} \\ \vdots & \ddots & \vdots \\ G_{n1}(s)e^{-\theta_{n1}s} & \cdots & G_{nn}(s)e^{-\theta_{nn}s} \end{bmatrix}$$

where $G_{ij}(s)(i, j = 1, 2, ..., n)$ are scalar rational transfer functions and $\theta_{ij} \geq 0$ are time delays

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where $G_{ij}(s)(i, j = 1, 2, ..., n)$ are scalar rational transfer functions and $\theta_{ij} \geq 0$ are time delays

Decentralized control: In decentralized control only those diagonal elements of the plant are treated; non-diagonal elements are regarded as uncertainty

Along this line, it seems that the plant can be factorized into the following form:

$$\mathbf{G}(s) = \mathbf{G_D}(s)\mathbf{G_O}(s)$$

where

$$\begin{aligned}
\mathbf{G_D}(s) &= \begin{bmatrix} e^{-\theta_{11}s} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{-\theta_{nn}s} \end{bmatrix} \\
\mathbf{G_O}(s) &= \begin{bmatrix} G_{11}(s) & \cdots & G_{1n}(s)e^{-(\theta_{1n}-\theta_{11})s} \\ \vdots & \ddots & \vdots \\ G_{n1}(s)e^{-(\theta_{n1}-\theta_{nn})s} & \cdots & G_{nn}(s) \end{bmatrix}
\end{aligned}$$

Unfortunately, such a factorization is not feasible for controller design, because there may be predictions in the resulting controller

SISO control: To find a feasible factorization, let us review the SISO design first. Given a plant $G(s)$, the ideal controller is

$$Q_{opt}(s) = G^{-1}(s)$$

In this case, the closed-loop transfer function is 1. The performance is evidently optimal. However, such a controller is not physically realizable when $G(s)$ has a time delay

An alternative is to factorize the plant into two parts:

$$G(s) = G_D(s)G_O(s)$$

where $G_D(s)$ is the time delay and $G_O(s)$ is the delay-free part

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If $G_O(s)$ is NMP, a further factorizing should be made:

$$G_O(s) = G_N(s)G_{MP}(s)$$

where $G_N(s)$ is the all-pass part and $G_{MP}(s)$ is the MP part

The optimal controller can be taken as

$$Q_{opt}(s) = G^{-1}(s)G_D(s)G_N(s) = G_{MP}^{-1}(s)$$

This controller is proved to be the optimal realizable inverse of $G^{-1}(s)$. When $Q_{opt}(s)$ is improper, a filter will be introduced to make it proper

MIMO case

The factorization for the time delay part: Let the inverse of the plant be

$$\mathbf{G}^{-1}(s) = \begin{bmatrix} G^{11}(s)e^{-\theta^{11}s} & \cdots & G^{1n}(s)e^{-\theta^{1n}s} \\ \cdots & \ddots & \cdots \\ G^{n1}(s)e^{-\theta^{n1}s} & \cdots & G^{nn}(s)e^{-\theta^{nn}s} \end{bmatrix}$$

where $G^{ji}(s)e^{-\theta^{ji}s}(j, i = 1, 2, ..., n)$ are the elements of $\mathbf{G}^{-1}(s)$, and θ^{ji} are the maximum time delays that can be separated from each element.

It is conjectured that the H_2 optimal decoupling controller can be designed in a similar way to that used in the SISO system

To carry out such a design, the first step is to factorize the plant $\mathbf{G}(s)$ into two parts:

$$\mathbf{G}(s) = \mathbf{G_D}(s)\mathbf{G_O}(s)$$

where $\mathbf{G_D}(s)$ is the time delay part. $\mathbf{G_D}(s)$ is diagonal and should be chosen such that:

1. it counteracts the predictions in $\mathbf{G}^{-1}(s)$ so that $\mathbf{G_O}^{-1}(s) = \mathbf{G}^{-1}(s)\mathbf{G_D}(s)$ does not involve predictions;
2. no additional time delays are introduced in $\mathbf{G_O}^{-1}(s)$.

Because the elements of $\mathbf{G_D}(s)$ are time delays, no RHP zeros and poles are cancelled in forming $\mathbf{G_O}(s) = \mathbf{G_D}^{-1}(s)\mathbf{G}(s)$. If $\mathbf{G_O}(s)$ is MP, the optimal controller can be taken as

$$\mathbf{Q_{opt}}(s) = \mathbf{G_O}^{-1}(s)$$

Definition

Let $\theta_{li}(i = 1, 2, ..., n)$ be the largest prediction of the ith column of $\mathbf{G}^{-1}(s)$, that is, $\theta_{li} = \max_j \theta^{ji}, j = 1, 2, ..., n$. The H_2 diagonal factorization for the time delay is defined as

$$\mathbf{G_D}(s) = \begin{bmatrix} e^{-\theta_{l1}s} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{-\theta_{ln}s} \end{bmatrix}$$

In particular, for rational plants $\mathbf{G_D}(s) = \mathbf{I}$

The definition is the same as that in quasi-H_∞ decoupling control. This is because **any** factorization with time delays shorter than the ones in this factorization will not thoroughly counteract the predictions in $\mathbf{G}^{-1}(s)$; an irrealizable controller will be obtained

It is easy to verify that $\mathbf{G_D}^H(j\omega)\mathbf{G_D}(j\omega) = \mathbf{I}$ for all ω and $\mathbf{G_D}(0) = \mathbf{I}$. The property $\mathbf{G_D}^H(j\omega)\mathbf{G_D}(j\omega) = \mathbf{I}$ is the generalization of the concept all-pass

The factorization for the RHP zero part: $\mathbf{G_O}(s)$ may be NMP. In this case, $\mathbf{G_O}(s)$ has to be factorized into two parts:

$$\mathbf{G_O}(s) = \mathbf{G_N}(s)\mathbf{G_{MP}}(s)$$

where $\mathbf{G_N}(s)$ is a diagonal matrix and should be chosen so that:

1. it is all-pass;
2. it counteracts the RHP poles in $\mathbf{G_O}^{-1}(s)$ so that $\mathbf{G_{MP}}^{-1}(s) = \mathbf{G_O}^{-1}(s)\mathbf{G_N}(s)$ is stable;
3. no additional RHP zeros are introduced in $\mathbf{G_{MP}}^{-1}(s)$

The $\mathbf{G_{MP}}(s)$ obtained in such a way is the MP part of $\mathbf{G_O}(s)$

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The optimal controller can be taken as

$$\mathbf{Q_{opt}}(s) = \mathbf{G_{MP}}^{-1}(s)$$

Assume that $z_j (j = 1, 2, ..., r_z)$ are the unstable poles of $\mathbf{G_O}^{-1}(s)$.

Definition

Let $k_{ij} (i = 1, 2, ..., n)$ be the largest multiplicity of the unstable pole $z_j (j = 1, 2, ..., r_z)$ in the ith column of $\mathbf{G_O}^{-1}(s)$. The H_2 diagonal factorization for closed RHP zeros is

$$\mathbf{G_N}(s) = \begin{bmatrix} \prod_{j=1}^{r_z} \left(\frac{-s+z_j}{s+\bar{z}_j}\right)^{k_{1j}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \prod_{j=1}^{r_z} \left(\frac{-s+z_j}{s+\bar{z}_j}\right)^{k_{nj}} \end{bmatrix}$$

In particular, for MP plants $\mathbf{G_N}(s) = \mathbf{I}$

It can be verified that $\mathbf{G_N}^H(j\omega)\mathbf{G_N}(j\omega) = \mathbf{I}$ for all ω and $\mathbf{G_N}(0) = \mathbf{I}$

The diagonal all-pass factorization: Let

$$\mathbf{G_A}(s) = \mathbf{G_D}(s)\mathbf{G_N}(s)$$

The diagonal all-pass factorization of $\mathbf{G}(s)$ can be expressed as

$$\mathbf{G}(s) = \mathbf{G_A}(s)\mathbf{G_{MP}}(s)$$

The factorization is unique and has the following features:

1. $\mathbf{G_A}(s)$ is diagonal;
2. $\mathbf{G_A}(s)$ is stable and all-pass;
3. $\mathbf{G_{MP}}(s)$ is MP

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3. $\mathbf{G_{MP}}(s)$ is MP

Why not Use The Inner-Outer Factorization?

Although the factorization defined in this section is similar to the inner-outer factorization, there are two different features between them:

1. The new factorization is **diagonal**
2. It is defined for both stable plants and **unstable** plants

13.3 H_2 Optimal Decoupling Control

The subject of this section: Derive the H_2 optimal decoupling controller by employing the controller parameterization in Section 13.1 and the diagonal factorization in Section 13.2 (Figure)

It will be shown that the conjecture in the last section about the optimal controller is correct. The diagonal factorization does result in the optimal solution

As discussed in Section 10.4, the H_2 optimal control problem can be expressed as

$$\min \|\mathbf{S}(s)\mathbf{W}(s)\|_2$$

where $\mathbf{W}(s) = \mathbf{I}/s$ is the weighting function and

$$\mathbf{S}(s) = \mathbf{I} - \mathbf{G}(s)\mathbf{Q}(s)$$

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It will be shown that the conjecture in the last section about the optimal controller is correct. The diagonal factorization does result in the optimal solution

As discussed in Section 10.4, the H_2 optimal control problem can be expressed as

$$\min \|\mathbf{S}(s)\mathbf{W}(s)\|_2$$

where $\mathbf{W}(s) = \mathbf{I}/s$ is the weighting function and

$$\mathbf{S}(s) = \mathbf{I} - \mathbf{G}(s)\mathbf{Q}(s)$$

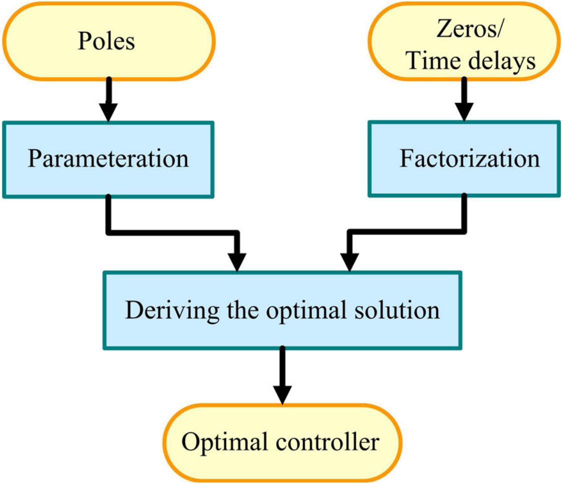

Figure: Design procedure for H_2 decoupling controller

For decoupling control, **$S(s)$ should be diagonal**. Since a fixed weighting function is adopted, the designer is not required to select a weighting function

The design problem here can be considered as an optimization over the class of all stabilizing controllers with asymptotic tracking properties

Preliminaries: Let H_2 be the set of all strictly proper stable functions, $H_2^{\perp}$ be the set of strictly proper transfer function matrices without poles in the open LHP. Then $L_2 := H_2 + H_2^{\perp}$ denotes the set of all strictly proper transfer function matrices without poles on the imaginary axis

Given a $\mathbf{T}(s)$ in L_2, it can be uniquely expressed as

$$\mathbf{T}(s) = \mathbf{T_1}(s) + \mathbf{T_2}(s)$$

where $\mathbf{T_1}(s) \in H_2$ and $\mathbf{T_2}(s) \in H_2^{\perp}$

Let the superscript * denote the **conjugate transpose of a system**: $\mathbf{T}^*(s) = \mathbf{T}^T(-s)$. The reader should not confuse it with the **complex conjugate transpose**, which is defined for a complex matrix: $\mathbf{T}^H(j\omega) = \bar{\mathbf{T}}^T(j\omega)$. For complex matrices, $\mathbf{T}^*(j\omega) = \mathbf{T}^H(j\omega)$

Lemma

If $\mathbf{T_1}(s) \in H_2$ *and* $\mathbf{T_2}(s) \in H_2^{\perp}$*, then*

$$\|\mathbf{T_1}(s) + \mathbf{T_2}(s)\|_2^2 = \|\mathbf{T_1}(s)\|_2^2 + \|\mathbf{T_2}(s)\|_2^2$$

Proof.

$$
\begin{aligned}
&\|\mathbf{T_1}(s)+\mathbf{T_2}(s)\|_2^2 \\
&= \frac{1}{2\pi}\int \mathrm{Trace}\{[\mathbf{T_1}(j\omega)+\mathbf{T_2}(j\omega)]^H[\mathbf{T_1}(j\omega)+\mathbf{T_2}(j\omega)]\}d\omega \\
&= \|\mathbf{T_1}(s)\|_2^2+\|\mathbf{T_2}(s)\|_2^2+ \\
&\quad \frac{1}{2\pi}\int \mathrm{Trace}[\mathbf{T_1}^H(j\omega)\mathbf{T_2}(j\omega)+\mathbf{T_2}^H(j\omega)\mathbf{T_1}(j\omega)]d\omega
\end{aligned}
$$

Consider the last term. Convert it into a contour integral by closing the imaginary axis with an infinite radius semicircle in the LHP:

$$
\begin{aligned}
&\frac{1}{2\pi}\int \mathrm{Trace}[\mathbf{T_1}^H(j\omega)\mathbf{T_2}(j\omega)+\mathbf{T_2}^H(j\omega)\mathbf{T_1}(j\omega)]d\omega \\
&= \frac{1}{2\pi j}\oint \mathrm{Trace}[\mathbf{T_1}^*(s)\mathbf{T_2}(s)+\mathbf{T_2}^*(s)\mathbf{T_1}(s)]ds
\end{aligned}
$$

Proof ctd.1.

According to Cauchy's theorem, if a function does not have poles in a bounded open set, then its integral on a closed contour in the set equals zero. Therefore, the right-hand side of the above equation equals zero $\square$

Design problem: Using the controller parameterization developed in Section 13.1, we have

$$
\begin{aligned}
&\|\mathbf{S}(s)\mathbf{W}(s)\|_2^2 \\
&= \left\|s^{-1}[\mathbf{I} - \mathbf{G}(s)\mathbf{Q}(s)]\right\|_2^2 \\
&= \left\|s^{-1}[\mathbf{I} - \mathbf{G}(s)\mathbf{G}^{-1}(0) - s\mathbf{G}(s)\mathbf{G}^{-1}(0)\mathbf{Q}_1(s)]\right\|_2^2
\end{aligned}
$$

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\end{aligned}
$$

Theorem

Assume that the plant with time delays can be uniquely factorized into two parts according to Definitions in Section 132:

$$\mathbf{G}(s) = \mathbf{G_A}(s)\mathbf{G_{MP}}(s)$$

where $\mathbf{G_A}(s) = \mathbf{G_D}(s)\mathbf{G_N}$. *Then the optimal solution for* H_2 *decoupled control is*

$$\mathbf{Q_{opt}}(s) = \mathbf{G_{MP}}^{-1}(s)$$

Proof.

An important property of the all-pass function is that it does not affect the value of 2-norm

Proof ctd.1.

that is,

$$\|\mathbf{G_A}(s)\mathbf{G_1}(s)\|_2 = \|\mathbf{G_1}(s)\|_2$$

for a transfer function matrix $\mathbf{G_1}(s)$. By utilizing the diagonal factorization given in the last section, the H_2 performance index can be written as

$$\begin{aligned}&\|\mathbf{S}(s)\mathbf{W}(s)\|_2^2\\ &= \left\|s^{-1}[\mathbf{G_A}^{-1}(s) - \mathbf{I}] + s^{-1}\{\mathbf{I} - \mathbf{G_{MP}}(s)\mathbf{G}^{-1}(0)[\mathbf{I} + s\mathbf{Q_1}(s)]\}\right\|_2^2\end{aligned}$$

Since $\mathbf{G_A}(0) = \mathbf{I}$ and $\mathbf{G_{MP}}(0)\mathbf{G}^{-1}(0) = \mathbf{I}$, s must be a factor of $\mathbf{G_A}^{-1}(s) - \mathbf{I}$ and $\mathbf{I} - \mathbf{G_{MP}}(s)\mathbf{G}^{-1}(0)$. $s^{-1}[\mathbf{G_A}^{-1}(s) - \mathbf{I}]$ is strictly proper. $s^{-1}\{\mathbf{I} - \mathbf{G_{MP}}(s)\mathbf{G}^{-1}(0)[\mathbf{I} + s\mathbf{Q_1}(s)]\}$ is also strictly proper if $\mathbf{Q}(s) = \mathbf{G}^{-1}(0)[\mathbf{I} + s\mathbf{Q_1}(s)]$ is proper

Proof ctd.2.

On the other hand, in light of the definition of $\mathbf{G_A}(s)$ $s^{-1}[\mathbf{G_A}^{-1}(s) - \mathbf{I}]$ is unstable. $s^{-1}\{\mathbf{I} - \mathbf{G_{MP}}(s)\mathbf{G}^{-1}(0)[\mathbf{I} + s\mathbf{Q_1}(s)]\}$ is also stable. To see this, let us consider the following equality:

$$\begin{aligned} & \mathbf{G_A}(s)\{\mathbf{I} - \mathbf{G_{MP}}(s)\mathbf{G}^{-1}(0)[\mathbf{I} + s\mathbf{Q_1}(s)]\} \\ = \ & (\mathbf{G_A}(s) - \mathbf{I}) + \{\mathbf{I} - \mathbf{G}(s)\mathbf{G}^{-1}(0)[\mathbf{I} + s\mathbf{Q_1}(s)]\} \end{aligned}$$

As $\mathbf{G_A}(s)$ is stable, the first term in the right-hand side is stable. By Corollary for decoupling controller parameterization, the second term is stable, too.
Applying Lemma, we have

$$\begin{aligned} & \|\mathbf{S}(s)\mathbf{W}(s)\|_2^2 \\ & = \left\|s^{-1}[\mathbf{G_A}^{-1}(s) - \mathbf{I}]\right\|_2^2 + \left\|s^{-1}\{\mathbf{I} - \mathbf{G_{MP}}(s)\mathbf{G}^{-1}(0)[I + s\mathbf{Q_1}(s)]\}\right\|_2^2 \end{aligned}$$

Proof ctd.3.

Minimizing the right-hand side of the equation yields

$$\mathbf{Q}_{\mathbf{1opt}}(s) = s^{-1}\mathbf{G}(0)\mathbf{G}_{\mathbf{MP}}{}^{-1}(s)[\mathbf{I} - \mathbf{G}_{\mathbf{MP}}(s)\mathbf{G}^{-1}(0)]$$

Hence, the optimal solution is

$$\mathbf{Q}_{\mathbf{opt}}(s) = \mathbf{G}_{\mathbf{MP}}{}^{-1}(s)$$

□

It can be seen from the deriving procedure that it is the parameterization in Section 13.1 and the factorization in Section 13.2 that make it possible to obtain an analytical solution. With the optimal analytical solution, the optimal performance is readily obtained as follows:

$$\min \|\mathbf{S}(s)\mathbf{W}(s)\|_2 = \left\|s^{-1}[\mathbf{G}_{\mathbf{A}}{}^{-1}(s) - \mathbf{I}]\right\|_2$$

Proof ctd.3.

Minimizing the right-hand side of the equation yields

$$\mathbf{Q_{1opt}}(s) = s^{-1}\mathbf{G}(0)\mathbf{G_{MP}}^{-1}(s)[\mathbf{I} - \mathbf{G_{MP}}(s)\mathbf{G}^{-1}(0)]$$

Hence, the optimal solution is

$$\mathbf{Q_{opt}}(s) = \mathbf{G_{MP}}^{-1}(s)$$

$\square$

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$$\min \|\mathbf{S}(s)\mathbf{W}(s)\|_2 = \left\|s^{-1}[\mathbf{G_A}^{-1}(s) - \mathbf{I}]\right\|_2$$

No matter what method is used, this is the best decoupling performance that can be achieved, provided the performance index is specified to minimize the 2-norm of the weighted sensitivity function

It should be emphasized that the proving procedure is general enough to allow general weighting functions, although only the pre-specified weighting function is considered in this section

Sometimes, the decoupled response can also achieve the optimal performance in a general sense, rather than diagonal optimal performance. **When the time delays in each row of $G(s)$ are the same, and all RHP zeros have the same multiplicities in each row of $G(s)$, the diagonal optimal solution is identical to the optimal solution**. Evidently, for MP plants, the results of the optimal control and the decoupling optimal control are the same

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13.4 Analysis for H_2 Decoupling Control Systems

The design procedure the MIMO H_2 controller design is similar to that for the SISO H_2 controller design. The next step is to introduce a filter $\mathbf{J}(s)$ to the optimal controller:

$$\mathbf{Q}(s) = \mathbf{Q}_{\mathbf{opt}}(s)\mathbf{J}(s)$$

The main functions of the filter:

1. The optimal controller $\mathbf{Q}_{\mathbf{opt}}(s)$ is usually improper. The filter is introduced to make it proper
2. The filter is used to tune the shape of the closed-loop response and satisfy the performance and robustness requirements, which was achieved by weights in many other methods

The filter should satisfy the following requirements:

1. The closed-loop system is internally stable
2. The controller $\mathbf{Q}(s)$ is proper
3. Asymptotic tracking

Since the closed-loop response is decoupled, $\mathbf{J}(s)$ is chosen as a diagonal matrix:

$$\mathbf{J}(s) = \begin{bmatrix} J_1(s) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & J_n(s) \end{bmatrix}$$

with

$$J_i(s) = \frac{N_{xi}(s)}{(\lambda_i s + 1)^{n_i}}, i = 1, 2, ..., n$$

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with

$$J_i(s) = \frac{N_{xi}(s)}{(\lambda_i s + 1)^{n_i}}, i = 1, 2, ..., n$$

Here $\lambda_i (i = 1, 2, ..., n)$ are performance degrees. $N_{xi}(s)$ are polynomials with all roots in the LHP.

It is not difficult to satisfy the first requirement on the filter. Let the ith element of $\mathbf{G_A}(s)$ be $G_{Ai}(s)(i = 1, 2, ..., n)$. By Corollary for decoupling controller parameterization, the closed-loop system is internally stable if

$$\lim_{s \to p_j} \frac{d^k}{ds^k}[1 - G_{Ai}(s)J_i(s)] = 0,$$

$$i = 1, 2, ..., n; j = 1, 2, ..., r_p; k = 0, 1, ..., l_{ij} - 1$$

Here $\deg\{N_{xi}(s)\} = \sum_{j=1}^{r_p} l_{ij} (i = 1, 2, ..., n)$

Assume that the largest relative degree of all elements in the ith column of $\mathbf{Q_{opt}}(s)$ is α_i. The second requirement can be satisfied by choosing $n_i = \deg\{N_{xi}(s)\} + \alpha_i$ for strictly proper columns and $n_i = \deg\{N_{xi}(s)\} + 1$ for semi-proper columns

To track the input asymptotically, the filter should satisfy

$$\mathbf{J}(0) = \mathbf{I}$$

This implies that the third requirement can be satisfied by choosing $N_{xi}(0) = 1$

Tuning: Each element of $\mathbf{J}(s)$ has an adjustable performance degree λ_i, which should be determined by the design specification. Since the closed-loop response is decoupled, each channel can be independently tuned

The ith element of the closed-loop transfer function matrix is

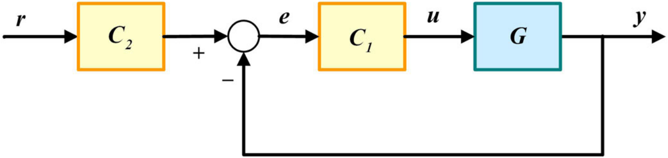
$$T_i(s) = \frac{N_{xi}(s)}{(\lambda_i s + 1)^{n_i}} \prod_{j=1}^{r_z} \left(\frac{-s + z_j}{s + \bar{z}_j} \right)^{k_{ij}} e^{-\theta_{ii} s}, i = 1, 2, ..., n$$

Since the performance degree is related to the nominal closed-loop response monotonically, the system can be tuned conveniently for quantitative responses. When there exists uncertainty, every channel is mainly affected by the performance degree of this channel. The tuning procedure is similar.

For stable plants, the controller can be implemented in the IMC structure. An important advantage of the implementation is that **the decoupling property can be thoroughly preserved** even when the rational approximation is used. For unstable plants, the controller must be implemented in the unity feedback loop

2DOF Control

Control systems with unstable plant usually exhibit excessive overshoot. This problem can be well solved by employing a 2DOF structure shown in Figure



A 2DOF controller can isolate the disturbance response from the reference response and thus make a better control possible

It is easy to design the 2DOF controller in the framework of this section

2DOF Control

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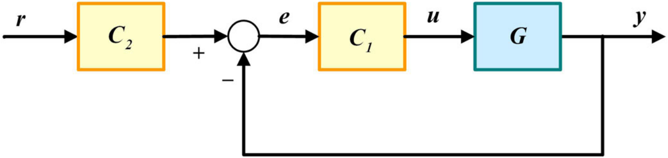

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It is easy to design the 2DOF controller in the framework of this section

The controller of the disturbance loop is just the controller of the unity feedback loop:

$$\mathbf{C}_1(s) = \mathbf{Q}(s)[\mathbf{I} - \mathbf{G}(s)\mathbf{Q}(s)]^{-1}$$

with $\mathbf{Q}(s) = \mathbf{Q}_{\mathbf{opt}}(s)\mathbf{J}_1(s)$. The closed-loop transfer function matrix of the corresponding unity feedback loop is

$$\mathbf{T}(s) = \mathbf{G}(s)\mathbf{Q}(s) = \mathbf{G}_{\mathbf{D}}(s)\mathbf{G}_{\mathbf{N}}(s)\mathbf{J}_1(s)$$

Regard $\mathbf{T}(s)$ as a new plant and $\mathbf{C}_2(s)$ as the IMC controller. The following optimal controller for the reference loop is obtained:

$$\mathbf{C}_{\mathbf{2opt}}(s) = \mathbf{J}_1{}^{-1}(s)\mathbf{G}_{\mathbf{N}}{}^{-1}(s)$$

Introduce a diagonal filter $\mathbf{J}_2(s)$ to the optimal controller:

$$\mathbf{C}_2(s) = \mathbf{C}_{\mathbf{2opt}}(s)\mathbf{J}_2(s)$$

$\mathbf{J}_2(s)$ has a similar structure to $\mathbf{J}_1(s)$

13.5 Design Examples for H_2 Decoupling Control

The design procedure for H_2 decoupling controllers can be formulated as follows:

1. If the plant does not contain time delays (that is, $\mathbf{G_A}(s) = \mathbf{I}$), turn to 3
2. If the plant contains time delays, take the rational part $\mathbf{G_O}(s)$ as the nominal plant
3. If $\mathbf{G_O}(s)$ does not have zeros in the RHP (that is, $\mathbf{G_N}(s) = \mathbf{I}$), take its inverse as $\mathbf{Q_{opt}}(s)$ and turn to 5
4. If $\mathbf{G_O}(s)$ has zeros in the RHP, construct an all-pass transfer function matrix by using the factor that contains the zero (that is, $\mathbf{G_N}(s)$) and then remove the all-pass transfer function matrix. Take the inverse of the remainder as $\mathbf{Q_{opt}}(s)$
5. Introduce a filter $\mathbf{J}(s)$ to $\mathbf{Q_{opt}}(s)$, compute $\mathbf{C}(s)$ and remove the RHP zero-pole cancellation in $\mathbf{C}(s)$

In this section, two examples are provided to help the reader to understand the design procedure of H_2 decoupling control. In the first example, a constructed unstable plant is used, because it is not easy to find a proper real unstable plant for the design problem to be illustrated. The plant in the second example is a real one

Example

The plant is described by the following transfer function matrix:

$$\boxed{\mathbf{G}(s) = \frac{1}{(s+3)(s-1)} \begin{bmatrix} s-2 & 2(s-2) \\ 1 & s-1 \end{bmatrix}}$$

which is NMP and unstable. The plant has two two-multiplicity poles at $s = -3$ and $s = 1$, respectively and two zeros at $s = 2$ and $s = 3$, respectively

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Example (ctd.1)

In view of Definition in Section 13.2, $\mathbf{G}(s)$ is factorized into

$$\mathbf{G}(s) = \mathbf{G_D}(s)\mathbf{G_O}(s)$$

Since $\mathbf{G}(s)$ is rational, $\mathbf{G_D}(s) = \mathbf{I}$.
The inverse of $\mathbf{G_O}(s)$ is

$$\mathbf{G_O}^{-1}(s) = \frac{\begin{bmatrix} s-1 & -2(s-2) \\ -1 & s-2 \end{bmatrix}}{\frac{(s-2)(s-3)}{(s-1)(s+3)}}$$

It has unstable poles. Hence, $\mathbf{G_O}(s)$ has to be factorized as follows:

$$\mathbf{G_O}(s) = \mathbf{G_N}(s)\mathbf{G_{MP}}(s)$$

Example (ctd.2)

According to Definition in Section 13.2, we have

$$\mathbf{G_N}(s) = \begin{bmatrix} \frac{-(s-2)}{s+2} & 0 \\ 0 & 1 \end{bmatrix} \frac{-(s-3)}{s+3},$$

$$\mathbf{G_{MP}}(s) = \frac{-1}{(s-3)(s-1)} \begin{bmatrix} -(s+2) & -2(s+2) \\ 1 & s-1 \end{bmatrix}.$$

By Theorem in Section 13.3, the optimal controller is

$$\mathbf{Q_{opt}}(s) = \mathbf{G_{MP}}^{-1}(s) = \frac{\begin{bmatrix} s-1 & 2(s+2) \\ -1 & -(s+2) \end{bmatrix}}{\frac{s+2}{s-1}}$$

Since both of the largest multiplicities of the RHP poles in the first and the second row of $\mathbf{G}(s)$ are 1, the following filter is chosen:

Example (ctd.3)

$$\mathbf{J}(s) = \begin{bmatrix} \frac{\beta_1 s+1}{(\lambda_1 s+1)^2} & 0 \\ 0 & \frac{\beta_2 s+1}{(\lambda_2 s+1)^2} \end{bmatrix}$$

The suboptimal controller is

$$\mathbf{Q}(s) = \mathbf{Q}_{\mathbf{opt}}(s)\mathbf{J}(s) = \begin{bmatrix} \frac{(s-1)(\beta_1 s+1)}{(\lambda_1 s+1)^2} & \frac{2(s+2)(\beta_2 s+1)}{(\lambda_2 s+1)^2} \\ \frac{-(\beta_1 s+1)}{(\lambda_1 s+1)^2} & \frac{-(s+2)(\beta_2 s+1)}{(\lambda_2 s+1)^2} \end{bmatrix} \frac{s-1}{s+2}$$

Simple computation gives

$$\begin{aligned} \mathbf{S}(s) &= \mathbf{I} - \mathbf{G}(s)\mathbf{Q}(s) \\ &= \begin{bmatrix} 1 - \frac{(s-2)(s-3)(\beta_1 s+1)}{(s+2)(s+3)(\lambda_1 s+1)^2} & 0 \\ 0 & 1 - \frac{-(s-3)(\beta_2 s+1)}{(s+3)(\lambda_2 s+1)^2} \end{bmatrix} \end{aligned}$$

Example (ctd.4)

It is readily obtained that

$$\beta_1 = 6(\lambda_1 + 1)^2 - 1, \beta_2 = 2(\lambda_2 + 1)^2 - 1$$

The unity feedback controller is

$$\mathbf{C}(s) = \begin{bmatrix} \frac{(s-1)(s+3)\{[6(\lambda_1+1)^2-1]s+1\}}{s[\lambda_1^2 s^2-(5+10\lambda_1)s+60\lambda_1+36\lambda_1^2+20]} & \frac{2(s+3)\{[2(\lambda_2+1)^2-1]s+1\}}{s(\lambda_2^2 s+6\lambda_2^2+6\lambda_2+1)} \\ \frac{-(s+3)\{[6(\lambda_1+1)^2-1]s+1\}}{s[\lambda_1^2 s^2-(5+10\lambda_1)s+60\lambda_1+36\lambda_1^2+20]} & \frac{-(s+3)\{[2(\lambda_2+1)^2-1]s+1\}}{s(\lambda_2^2 s+6\lambda_2^2+6\lambda_2+1)} \end{bmatrix}$$

If $\lambda_1 = \lambda_2 = 1$, the controller is

$$\boxed{\mathbf{C}(s) = \begin{bmatrix} \frac{23s^3+47s^2-67s-3}{s(s^2-15s+116)} & \frac{14s^2+44s+6}{s(s+13)} \\ \frac{-(23s^2+70s+3)}{s(s^2-15s+116)} & \frac{-(7s^2+22s+3)}{s(s+13)} \end{bmatrix}}$$

The closed-loop responses are shown in Figures

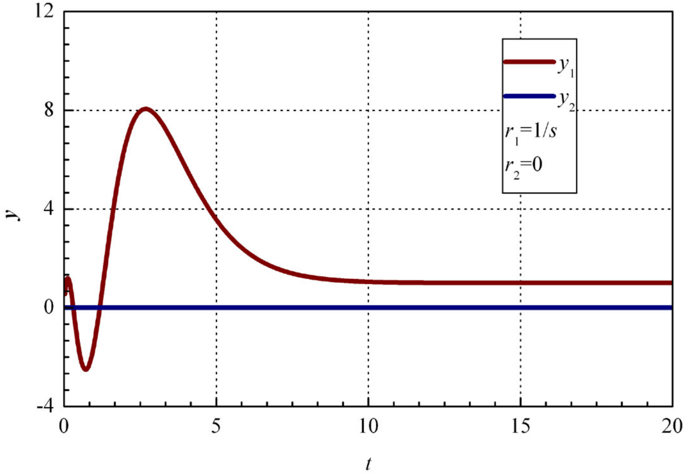


Figure: Closed-loop response with $\lambda_1 = \lambda_2 = 1\text{-}1$

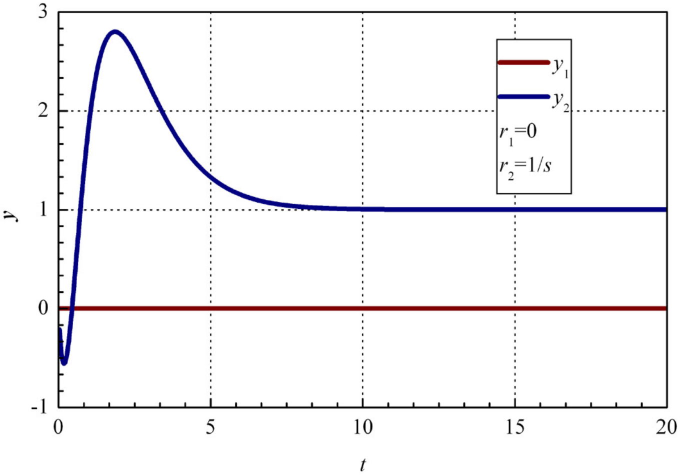


Figure: Closed-loop response with $\lambda_1 = \lambda_2 = $ 1-2

Example (ctd.5)

Because the plant is unstable, there are large overshoots in the reference response

If a 2DOF structure is adopted, the controller for the disturbance loop is

$$\boxed{\mathbf{C_1}(s) = \mathbf{C}(s)}$$

The controller for the reference loop is

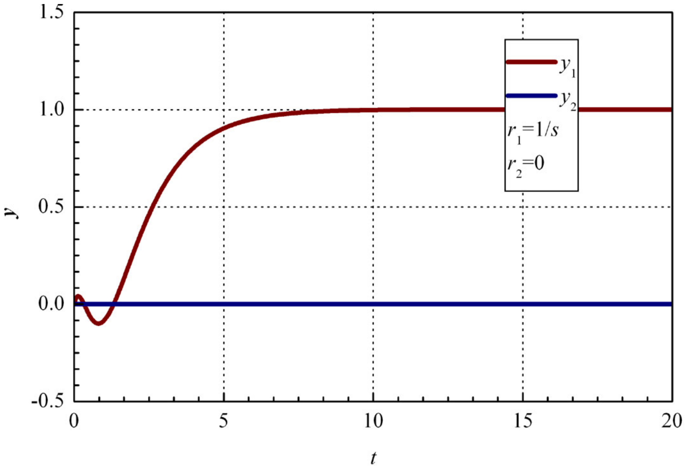
$$\mathbf{C_2}(s) = \begin{bmatrix} \frac{(\lambda_1 s+1)^2}{(\beta_1 s+1)(\lambda_1' s+1)} & 0 \\ 0 & \frac{(\lambda_2 s+1)^2}{(\beta_2 s+1)(\lambda_2' s+1)} \end{bmatrix}$$

Example (ctd.6)

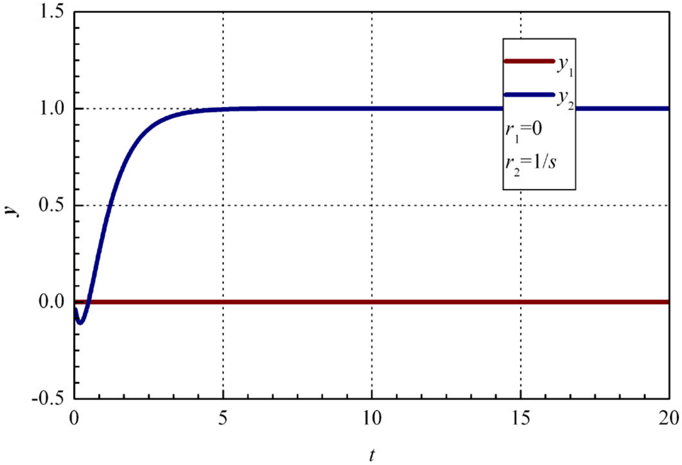
For 10% undershoot in each reference loop, $\lambda_1' = 1.4$ and $\lambda_2' = 0.8$. The controller is

$$\boxed{\mathbf{C_2}(s) = \begin{bmatrix} \frac{s^2+2s+1}{32.2s^2+24.4s+1} & 0 \\ 0 & \frac{s^2+2s+1}{5.6s^2+7.8s+1} \end{bmatrix}}$$

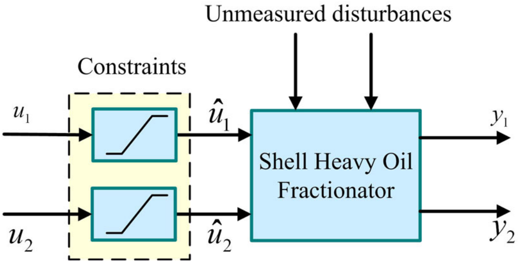
The closed-loop responses are shown in Figures. Now there are not overshoots in the reference response



Closed-loop response with $\lambda_1' = 1.4$ and $\lambda_2' = 0.8$-1



Closed-loop response with $\lambda_1' = 1.4$ and $\lambda_2' = 0.8$-2



Example

Consider a heavy oil fractionator (Figure), of which the linearized model is

$$\boxed{\mathbf{G}(s) = \begin{bmatrix} \frac{4.05e^{-27s}}{27s+1} & \frac{1.77e^{-28s}}{60s+1} \\ \frac{5.39e^{-18s}}{50s+1} & \frac{5.72e^{-14s}}{60s+1} \end{bmatrix}}$$

Example (ctd.1)

in which the time constants and time delays are expressed in minutes. The main objective is to maintain process outputs y_1 and y_2 at specification 0.0 ± 0.005 in the steady state, while at the same time the process inputs u_1 and u_2 are subject to saturation ± 0.5 with respect to every unit step reference.

The inverse of the plant is

$$\mathbf{G}^{-1}(s) = \frac{\mathrm{adj}[\mathbf{G}(s)]}{\det[\mathbf{G}(s)]}$$

where

$$\mathrm{adj}[\mathbf{G}(s)] = \begin{bmatrix} \frac{5.72e^{-14s}}{60s+1} & -\frac{1.77e^{-28s}}{60s+1} \\ -\frac{5.39e^{-18s}}{50s+1} & \frac{4.05e^{-27s}}{27s+1} \end{bmatrix}$$

Example (ctd.2)

$$\det[\mathbf{G}(s)] = \left(\frac{4.05}{27s+1}\frac{5.72}{60s+1} - \frac{1.77}{60s+1}\frac{5.39e^{-5s}}{50s+1}\right)e^{-41s}$$

It can be verified that

$$\begin{aligned}\mathbf{G_D}(s) &= \begin{bmatrix} e^{-27s} & 0 \\ 0 & e^{-14s} \end{bmatrix} \\ \mathbf{G_N}(s) &= \mathbf{I}\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbf{G_O}(s) &= \mathbf{G_D}^{-1}(s)\mathbf{G}(s) = \begin{bmatrix} \frac{4.05}{27s+1} & \frac{1.77e^{-s}}{60s+1} \\ \frac{5.39e^{-4s}}{50s+1} & \frac{5.72}{60s+1} \end{bmatrix} \\ \mathbf{G_{MP}}(s) &= \mathbf{G_N}^{-1}(s)\mathbf{G_O}(s) = \mathbf{G_O}(s)\end{aligned}$$

Example (ctd.3)

A little algebra gives

$$\mathbf{Q}_{\mathbf{opt}}(s) = \mathbf{G}_{\mathbf{MP}}{}^{-1}(s) = \begin{bmatrix} \frac{4.05}{27s+1} & \frac{1.77e^{-s}}{60s+1} \\ \frac{5.39e^{-4s}}{50s+1} & \frac{5.72}{60s+1} \end{bmatrix}^{-1}$$

$$= \frac{\begin{bmatrix} \frac{5.72}{60s+1} & -\frac{1.77e^{-s}}{60s+1} \\ -\frac{5.39e^{-4s}}{50s+1} & \frac{4.05}{27s+1} \end{bmatrix}}{\frac{4.05}{27s+1}\frac{5.72}{60s+1} - \frac{1.77}{60s+1}\frac{5.39e^{-5s}}{50s+1}}$$

This rigorously analytical controller is of infinite dimension. With the help of fitting techniques, one obtains

$$\frac{4.05}{27s+1}\frac{5.72}{60s+1} - \frac{1.77}{60s+1}\frac{5.39e^{-5s}}{50s+1} \approx \frac{13.6257}{1193.2s^2 + 67.4s + 1}$$

For step inputs, choose the following filter:

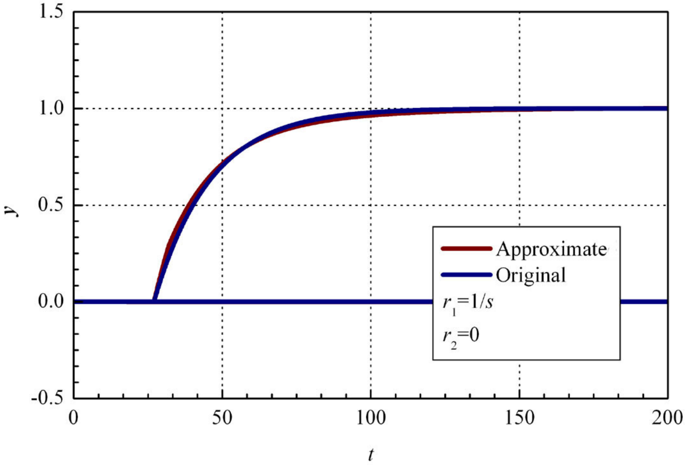
Example (ctd.4)

$$\mathbf{J}(s) = \begin{bmatrix} \frac{1}{\lambda_1 s+1} & 0 \\ 0 & \frac{1}{\lambda_2 s+1} \end{bmatrix}$$

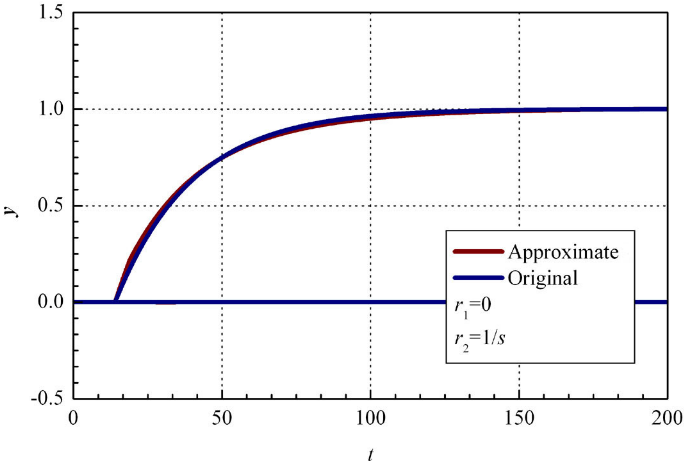
The controller is

$$\boxed{\mathbf{Q}(s) = (1193.2\mathrm{s}^2 + 67.4s + 1) \begin{bmatrix} \frac{0.4198}{(60s+1)(\lambda_1 s+1)} & -\frac{0.1299e^{-s}}{(60s+1)(\lambda_2 s+1)} \\ -\frac{0.3956e^{-4s}}{(50s+1)(\lambda_1 s+1)} & \frac{0.2972}{(27s+1)(\lambda_2 s+1)} \end{bmatrix}}$$

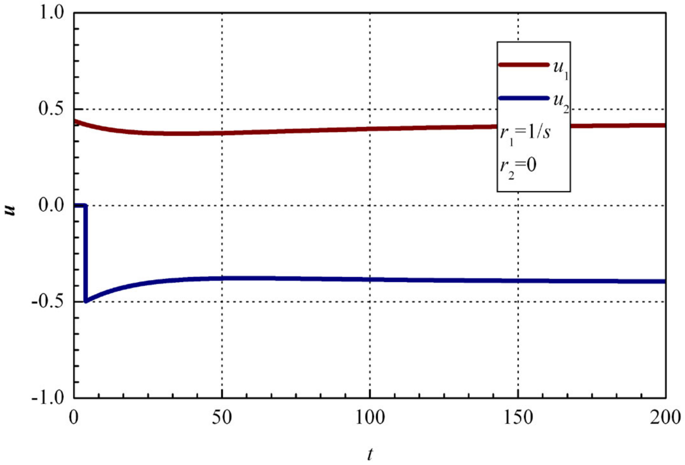
Increase the performance degrees from small to large. It is found that $\lambda_1 = 19$ and $\lambda_2 = 26$ can provide the required response. The closed-loop responses and the manipulated variable responses are shown in Figures. Since the IMC structure is used, the closed-loop responses are thoroughly decoupled even for the approximate controller



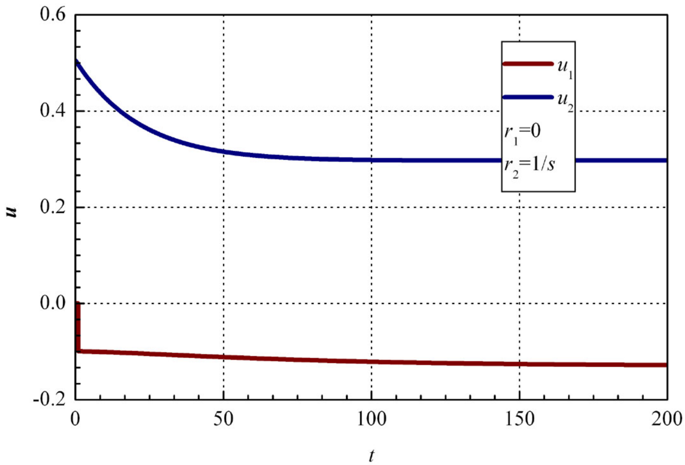
Closed-loop response with $\lambda_1 = 19$ and $\lambda_2 = 26\text{-}1$



Closed-loop response with $\lambda_1 = 19$ and $\lambda_2 = 26$-2



Response of manipulated variables for $\lambda_1 = 19$ and $\lambda_2 = 26$-1



Response of manipulated variables for $\lambda_1 = 19$ and $\lambda_2 = 26$-2

End of Chapter 13