Chapter 7

Digraphs

There are occasions when the symmetric nature of graphs does not provide a desirable structure to represent a situation we may encounter. This leads us to the concept of directed graphs (digraphs).

7.1 Introduction to Digraphs

A directed graph or digraph $D$ is a finite nonempty set of objects called vertices together with a (possibly empty) set of ordered pairs of distinct vertices of $D$ called arcs or directed edges. For vertices $u$ and $v$ in $D$, an arc $(u, v)$ is sometimes denoted by writing $u \rightarrow v$ (or $v \leftarrow u$). As with graphs, the vertex set of $D$ is denoted by $V(D)$ or simply $V$ and the arc set (or directed edge set) of $D$ is denoted by $E(D)$ or $E$. A digraph $D$ with vertex set $V = \{u, v, w, x\}$ and arc set $E = \{(u, v), (v, u), (u, w), (w, v), (w, x)\}$ is shown in Figure 7.1. When a digraph is described by means of a diagram, the “direction” of each arc is indicated by an arrowhead. Observe that in a digraph, it is possible for two arcs to join the same pair of vertices if the arcs are directed oppositely.

![Figure 7.1: A digraph](image)

Much of the terminology used for digraphs is quite similar to that used for graphs. The cardinality of the vertex set of a digraph $D$ is called the order of $D$ and is ordinarily denoted by $n$, while the cardinality of its arc set is the size
of $D$ and is ordinarily denoted by $m$. If $a = (u, v)$ is an arc of a digraph $D$, then $u$ is said to be adjacent to $v$ and $v$ is adjacent from $u$. For a vertex $v$ in a digraph $D$, the outdegree $\text{od} v$ of $v$ is the number of vertices of $D$ to which $v$ is adjacent, while the indegree $\text{id} v$ of $v$ is the number of vertices of $D$ from which $v$ is adjacent. The out-neighborhood $N^+(v)$ of a vertex $v$ in a digraph $D$ is the set of vertices adjacent from $v$, while the in-neighborhood $N^-(v)$ of $v$ is the set of vertices adjacent to $v$. Thus, $\text{od} v = |N^+(v)|$ and $\text{id} v = |N^-(v)|$.

The degree $\text{deg} v$ of a vertex $v$ is defined by

$$\text{deg} v = \text{od} v + \text{id} v.$$ 

For the vertex $v$ in the digraph of Figure 7.2, $\text{od} v = 3$, $\text{id} v = 2$ and $\text{deg} v = 5$.

Figure 7.2: The outdegree, indegree and degree of a vertex

**The First Theorem of Digraph Theory**

The directed graph version of Theorem 1.4 is stated next.

**Theorem 7.1 (The First Theorem of Digraph Theory)** If $D$ is a digraph of size $m$, then

$$\sum_{v \in V(G)} \text{od} v = \sum_{v \in V(G)} \text{id} v = m.$$ 

**Proof.** When the outdegrees of the vertices are summed, each arc is counted once. Similarly, when the indegrees of the vertices are summed, each arc is counted just once. 

A digraph $D_1$ is isomorphic to a digraph $D_2$, written $D_1 \cong D_2$, if there exists a bijective function $\phi : V(D_1) \to V(D_2)$ such that $(u, v) \in E(D_1)$ if and only if $(\phi(u), \phi(v)) \in E(D_2)$. The function $\phi$ is called an isomorphism from $D_1$ to $D_2$.

There is only one digraph of order 1, namely the trivial digraph. Also, there is only one digraph of order 2 and size $m$ for each $m$ with $0 \leq m \leq 2$. There are four digraphs of order 3 and size 3, all of which are shown in Figure 7.3.
7.1. INTRODUCTION TO DIGRAPHS

A digraph $D_1$ is a subdigraph of a digraph $D$ if $V(D_1) \subseteq V(D)$ and $E(D_1) \subseteq E(D)$. We use $D_1 \subseteq D$ to indicate that $D_1$ is a subdigraph of $D$. A subdigraph $D_1$ of $D$ is a spanning subdigraph of $D$ if $V(D_1) = V(D)$. Vertex-deleted, arc-deleted, induced and arc-induced subdigraphs are defined in the expected manner. These last two concepts are illustrated for the digraph $D$ of Figure 7.4, where

$$V(D) = \{v_1, v_2, v_3, v_4\}, \quad U = \{v_1, v_2, v_3\} \quad \text{and} \quad X = \{(v_1, v_2), (v_2, v_4)\}.$$ 

![Figure 7.3: The digraphs of order 3 and size 3](image)

Figure 7.3: The digraphs of order 3 and size 3

We now consider certain types of digraphs that occur periodically. A digraph is symmetric if whenever $(u, v)$ is an arc of $D$, then $(v, u)$ is an arc of $D$ as well. There is a natural one-to-one correspondence between symmetric digraphs and graphs. The complete symmetric digraph $K_n^s$ of order $n$ has both arcs $(u, v)$ and $(v, u)$ for every two distinct vertices $u$ and $v$. A digraph is called an oriented graph if whenever $(u, v)$ is an arc of $D$, then $(v, u)$ is not an arc of $D$. Thus, an oriented graph $D$ can be obtained from a graph $G$ by assigning a direction to (or by “orienting”) each edge of $G$, thereby transforming each edge of a graph $G$ into an arc and transforming $G$ itself into an oriented graph. The digraph $D$ is also called an orientation of $G$. Figure 7.5 shows three digraphs $D_1$, $D_2$ and $D_3$. While $D_1$ is a symmetric digraph and $D_2$ is an oriented graph, the digraph $D_3$ is neither. The underlying graph of a digraph $D$ is that graph obtained by replacing each arc $(u, v)$ or symmetric pair $(u, v), (v, u)$ of arcs by the edge $uv$. The underlying graph of each digraph in Figure 7.5 is the graph $G$. 

![Figure 7.4: Induced and arc-induced subdigraphs](image)

Figure 7.4: Induced and arc-induced subdigraphs
Theorem 7.2

If \( \text{od} v \geq k \geq 1 \) for every vertex \( v \) of \( D \), then \( D \) contains a cycle of length at least \( k + 1 \).
7.1. INTRODUCTION TO DIGRAPHS

Connected Digraphs

A digraph $D$ is connected (or weakly connected) if the underlying graph of $D$ is connected. A digraph $D$ is strong (or strongly connected) if for every pair $u, v$ of vertices, $D$ contains both a $u - v$ path and a $v - u$ path. While all digraphs of Figure 7.7 are connected, only $D_1$ is strong.

![Figure 7.7: Connectedness properties of digraphs](image)

Distance can be defined in digraphs as well. For vertices $u$ and $v$ in a digraph $D$ containing a $u - v$ path, the directed distance $\bar{d}(u, v)$ from $u$ to $v$ is the length of a shortest $u - v$ path in $D$. The distances $\bar{d}(u, v)$ and $\bar{d}(v, u)$ are defined for all pairs $u, v$ of vertices in a digraph $D$ if and only if $D$ is strong. This distance is not a metric, in general. Although directed distance satisfies the triangle inequality, it is not symmetric unless $D$ is symmetric, in which case $D$ can be considered a graph. Eccentricity can be defined as before, as well as radius and diameter in a strong digraph $D$. The eccentricity $e(u)$ of a vertex $u$ in $D$ is the distance from $u$ to a vertex farthest from $u$. The minimum eccentricity of the vertices of $D$ is the radius $\text{rad}(D)$ of $D$, while the diameter $\text{diam}(D)$ is the greatest eccentricity.

Each vertex of the strong digraph $D$ of Figure 7.8 is labeled with its eccentricity. Observe that $\text{rad}(D) = 2$ and $\text{diam}(D) = 5$, so it is not true, in general, that $\text{diam}(D) \leq 2 \text{rad}(D)$, as is the case with graphs.

![Figure 7.8: Eccentricities in a strong digraph](image)
CHAPTER 7. DIGRAPHS

7.2 Strong Digraphs

We saw that there are two types of connectedness for digraphs, namely weakly connected (or, more simply, connected) digraphs and strongly connected (or simply strong) digraphs. In this section, we explore strong digraphs in more detail.

The following theorem is the digraph analogue of Theorem 2.1 and its proof is analogous as well (see Exercise 15).

Theorem 7.3 Let $u$ and $v$ be two vertices in a digraph $D$. For every $u-v$ walk $W$ in $D$, there exists a $u-v$ path $P$ such that every arc of $P$ belongs to $W$.

Strong digraphs are characterized in the following theorem.

Theorem 7.4 A digraph $D$ is strong if and only if $D$ contains a closed spanning walk.

Proof. Assume that $W = (u_1, u_2, \ldots, u_k, u_1)$ is a closed spanning walk in $D$. Let $u, v \in V(D)$. Then $u = u_i$ and $v = u_j$ for some $i, j$ with $1 \leq i, j \leq k$ and $i \neq j$. Without loss of generality, assume that $i < j$. Then $W_1 = (u_i, u_{i+1}, \ldots, u_j)$ is a $u_i - u_j$ walk in $D$ and $W_2 = (u_j, u_{j+1}, \ldots, u_k, u_1, \ldots, u_i)$ is a $u_j - u_i$ walk in $D$. By Theorem 7.3, $D$ contains both a $u_i - u_j$ path and a $u_j - u_i$ path in $D$ and so $D$ is strong.

Conversely, assume that $D$ is a nontrivial strong digraph with $V(D) = \{v_1, v_2, \ldots, v_n\}$ and consider the cyclic sequence $v_1, v_2, \ldots, v_n, v_{n+1} = v_1$. Since $D$ is strong, $D$ contains a $v_i - v_{i+1}$ path $P_i$ for $i = 1, 2, \ldots, n$. Then the sequence $P_1, P_2, \ldots, P_n$ of paths produces a closed spanning walk in $D$.

The converse $\bar{D}$ of a digraph $D$ is obtained from $D$ by reversing the direction of every arc of $D$. Thus, $\bar{D}$ is strong if and only if its converse $\bar{D}$ is strong (see Exercise 16).

Robbins’ Theorem

We saw that an orientation of a graph $G$ is a digraph obtained by assigning a direction to each edge of $G$. Herbert Robbins (1922–2001) studied those graphs having a strong orientation. Certainly, if $G$ has a strong orientation, then $G$ must be connected. Also, if $G$ has a bridge, then it is impossible to produce a strong orientation of $G$. Robbins [205] showed that this is all that’s required for $G$ to have a strong orientation.

Theorem 7.5 (Robbins’ Theorem) A nontrivial graph $G$ has a strong orientation if and only if $G$ is connected and bridgeless.

Proof. We have already observed that if a graph $G$ has a strong orientation, then $G$ is connected and bridgeless. Suppose that the converse is false. Then
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there exists a connected and bridgeless graph $G$ that has no strong orientation. Among the subgraphs of $G$, let $H$ be one of maximum order that has a strong orientation. Such a subgraph exists since for each $v \in V(G)$, the subgraph $G[\{v\}]$ trivially has a strong orientation. Thus, $|V(H)| < |V(G)|$, since, by assumption, $G$ has no strong orientation.

Assign directions to the edges of $H$ so that the resulting digraph $D$ is strong, but assign no directions to the edges of $G - E(H)$. Let $u \in V(H)$ and let $v \in V(G) - V(H)$. Since $G$ is connected and bridgeless, it follows by Theorem 4.18 that $G$ contains two edge-disjoint $u - v$ paths. Let $P$ be one of these $u - v$ paths and let $Q$ be the $v - u$ path that results from the other $u - v$ path. Further, let $u_1$ be the last vertex of $P$ that belongs to $H$, and let $v_1$ be the first vertex of $Q$ belonging to $H$. Next, let $P_1$ be the $u_1 - v$ subpath of $P$ and let $Q_1$ be the $v - v_1$ subpath of $Q$. Direct the edges of $P_1$ from $u_1$ toward $v$, producing the directed path $P'_1$, and direct the edges of $Q_1$ from $v$ toward $v_1$, producing the directed path $Q'_1$.

Define the digraph $D'$ by

$$V(D') = V(D) \cup V(P'_1) \cup V(Q'_1) \quad \text{and} \quad E(D') = E(D) \cup E(P'_1) \cup E(Q'_1).$$

Since $D$ is strong, so too is $D'$, contradicting the choice of $H$.

As we mentioned, Theorem 7.5 is due to Robbins. The paper in which this theorem appears is titled “A theorem on graphs, with an application to a problem of traffic control” and was published in 1939 in the American Mathematical Monthly, only a year after Robbins received his Ph.D. from Harvard University in topology, under the direction of Hassler Whitney. This was only Robbins’ second publication of what would become a long and impressive list. Also in 1939, at age 24, Robbins began work on the classic book *What is Mathematics?* with Richard Courant. Robbins classified this book as a literary work rather than a scientific work. This book discussed mathematics as it existed at that time. A few years later, Robbins became interested in and devoted his research to mathematical statistics, to which he made major contributions.

7.3 Eulerian and Hamiltonian Digraphs

Eulerian and Hamiltonian graphs have natural analogues for digraphs. In both instances, these are strong digraphs.

**Eulerian Digraphs**

An **Eulerian circuit** in a connected digraph $D$ is a circuit that contains every arc of $D$ (necessarily exactly once); while an **Eulerian trail** in $D$ is an open trail that contains every arc of $D$. A connected digraph that contains an Eulerian circuit is an **Eulerian digraph**. The next theorem gives a character-
There are sufficient conditions for a digraph to be Hamiltonian, however, that are analogues of the simpler sufficient conditions for graphs to be Hamiltonian. Indeed, if anything, the situation for Hamiltonian digraphs is even more complex than it is for Hamiltonian graphs. The proofs of these results, unlike their graphical counterparts, are quite lengthy and, for this reason, are not given here.

The following result of Henri Meyniel [170] gives a sufficient condition (much like that in Theorem 6.4 for graphs) for a digraph to be Hamiltonian.
Theorem 7.8 (Meyniel’s Theorem) If $D$ is a nontrivial strong digraph of order $n$ such that
\[ \deg u + \deg v \geq 2n - 1 \]
for every pair $u, v$ of nonadjacent vertices, then $D$ is Hamiltonian.

Among the consequences of Theorem 7.8 is a result obtained by Douglas Woodall [259].

Corollary 7.9 If $D$ is a nontrivial digraph of order $n$ such that
\[ \od u + \id v \geq n \]
whenever $u$ and $v$ are distinct vertices with $(u, v) \notin E(D)$, then $D$ is Hamiltonian.

The proof of the following theorem (due to Alain Ghouila-Houri [104]) is an immediate consequence of Theorem 7.8.

Corollary 7.10 If $D$ is a strong digraph of order $n$ such that $\deg v \geq n$ for every vertex $v$ of $D$, then $D$ is Hamiltonian.

Corollary 7.10 also has a corollary. We provide a proof of this result.

Corollary 7.11 If $D$ is a digraph of order $n$ such that
\[ \od v \geq n/2 \quad \text{and} \quad \id v \geq n/2 \]
for every vertex $v$ of $D$, then $D$ is Hamiltonian.

Proof. Suppose that the theorem is false. Since the theorem is clearly true for $n = 2$ and $n = 3$, there exists some integer $n \geq 4$ and a digraph $D$ of order $n$ that satisfies the hypothesis but which is not Hamiltonian. Let $C$ be a cycle in $D$ of maximum length $k$. It follows from Theorem 7.2 and the assumption that $D$ is not Hamiltonian that $1 + n/2 \leq k < n$. Also, let $P$ be a path of maximum length such that no vertex of $P$ lies on $C$. Suppose that $P$ is a $u-v$ path of length $\ell \geq 0$. Therefore, $k + \ell + 1 \leq n$. (See Figure 7.10.)

Since
\[ \ell \leq n - k - 1 \leq n - \left(1 + \frac{n}{2}\right) - 1 = \frac{n}{2} - 2, \]
it follows that $\ell \leq n/2 - 2$ and that there are at least two vertices adjacent to $u$ which do not lie on $P$. Since $P$ is a longest path all of whose vertices do not lie on $C$, it follows that there are at least two vertices that lie on $C$ that are adjacent to $u$ and at least two vertices adjacent from $v$ which lie on $C$.

Let $a$ denote the number of vertices on $C$ that are adjacent to $u$. Thus, $a \geq 2$. For every vertex $x$ on $C$ that is adjacent to $u$, the $\ell + 1$ vertices immediately following $x$ on $C$ are not adjacent from $v$, for otherwise, $D$ has a cycle of length exceeding $k$. Since $C$ contains vertices adjacent from $v$, there must be a vertex
y on $C$ that is adjacent to $u$ such that none of the $\ell + 1$ vertices immediately following $y$ on $C$ are adjacent to $u$ or adjacent from $v$.

For each of the $a - 1$ vertices on $C$ that are distinct from $y$ and adjacent to $u$, the vertex immediately following it cannot be adjacent from $v$. Hence, at least $(a - 1) + (\ell + 1) = a + \ell$ vertices on $C$ are not adjacent from $v$, for otherwise again, $D$ has a cycle of length exceeding $k$. Since $P$ is a longest path in $D$ containing no vertices of $C$, every vertex adjacent to $u$ is either on $C$ or on $P$.

Because $\text{id}_u \geq n/2$ and the only vertices of $D$ that can be adjacent to $u$ belong to $C$ or $P$, it follows that $a + \ell \geq n/2$. Therefore, $v$ is adjacent to at most $(n - 1) - (a + \ell) \leq (n - 1) - n/2 = n/2 - 1$ vertices, producing a contradiction. 

7.4 Tournaments

There are sporting events involving teams (or individuals) that require every two teams to compete against each other exactly once. This is referred to as a round robin tournament. Men’s soccer has been part of the Summer Olympic Games since 1900. Teams from 16 countries participate, divided into four pools of four teams each. In each pool, a round robin tournament takes place, in which the top two teams in each pool advance to play for Olympic medals. This also occurs during the World Cup for soccer supremacy when 32 countries participate, divided into eight pools of four teams each.

Round robin tournaments give rise quite naturally to a class of digraphs that we mentioned earlier. Recall that a tournament is an orientation of a complete graph. Therefore, a tournament can be defined as a digraph such that for every pair $u, v$ of distinct vertices, exactly one of $(u, v)$ and $(v, u)$ is an arc. A tournament $T$ then models a round robin tournament in which no ties are permitted. The vertices of $T$ are the teams in the round robin tournament and $(u, v)$ is an arc in $T$ if team $u$ defeats team $v$. 
7.4. TOURNAMENTS

Figure 7.11 shows two tournaments of order 3. In fact, these are the only two tournaments of order 3. The number of non-isomorphic tournaments increases sharply with their orders. For example, there is only one tournament of order 1 and one of order 2. As we just observed, the tournaments $T_1$ and $T_2$ in Figure 7.11 are the only two tournaments of order 3. There are four tournaments of order 4, 12 of order 5, 56 of order 6 and over 154 billion of order 12.

\[ T_1 : \quad T_2 : \]

Figure 7.11: The tournaments of order 3

Since the size of a tournament of order $n$ is $\binom{n}{2}$, it follows from Theorem 7.1 that

\[ \sum_{v \in V(T)} \text{od} v = \sum_{v \in V(T)} \text{id} v = \binom{n}{2}. \]

Transitive Tournaments

A tournament $T$ is transitive if whenever $(u, v)$ and $(v, w)$ are arcs of $T$, then $(u, w)$ is also an arc of $T$. The tournament $T_2$ of Figure 7.11 is transitive while $T_1$ is not. The following result gives an elementary property of transitive tournaments. An acyclic digraph is a digraph having no cycles.

**Theorem 7.12** A tournament is transitive if and only if it is acyclic.

**Proof.** Let $T$ be an acyclic tournament and suppose that $(u, v)$ and $(v, w)$ are arcs of $T$. Since $T$ is acyclic, $(w, u) \notin E(T)$. Therefore, $(u, w) \in E(T)$ and $T$ is transitive.

Conversely, suppose that $T$ is a transitive tournament and assume that $T$ contains a cycle, say $C = (v_1, v_2, \ldots, v_k, v_1)$, where $k \geq 3$. Since $(v_1, v_2)$ and $(v_2, v_3)$ are arcs of the transitive tournament $T$, it follows that $(v_1, v_3)$ is also an arc of $T$. Since $(v_1, v_3)$ and $(v_3, v_4)$ are arcs, if $k \geq 4$, then $(v_1, v_4)$ is an arc. Similarly, $(v_1, v_5), (v_1, v_6), \ldots, (v_1, v_k)$ are arcs of $T$. However, this contradicts the fact that $(v_k, v_1)$ is an arc of $T$. Thus, $T$ is acyclic.

Suppose that a tournament $T$ of order $n$ with vertex set $V(T) = \{v_1, v_2, \ldots, v_n\}$ represents a round robin tournament involving competition among $n$ teams $v_1, v_2, \ldots, v_n$. If team $v_i$ defeats team $v_j$, then $(v_i, v_j)$ is an arc of $T$. The number of victories by team $v_i$ is the outdegree of $v_i$. For this reason, the outdegree of the vertex $v_i$ in a tournament is also referred to as the **score** of $v_i$. A sequence $s_1, s_2, \ldots, s_n$ of nonnegative integers is called a **score sequence of**
a tournament if there exists a tournament $T$ of order $n$ whose vertices can be labeled $v_1, v_2, \ldots, v_n$ such that $\text{od} \ v_i = s_i$ for $i = 1, 2, \ldots, n$.

Figure 7.12 shows transitive tournaments of order $n$ for $n = 3, 4, 5$. The score sequence of every transitive tournament has an interesting property. The following result describes precisely which sequences are score sequences of transitive tournaments.

![Figure 7.12: Transitive tournaments of orders 3, 4, 5](image)

**Theorem 7.13** A nondecreasing sequence $\pi$ of $n$ nonnegative integers is a score sequence of a transitive tournament of order $n$ if and only if $\pi$ is the sequence $0, 1, \ldots, n - 1$.

**Proof.** First we show that $\pi : 0, 1, \ldots, n - 1$ is a score sequence of a transitive tournament of order $n$. Let $T$ be the tournament with vertex set $V(T) = \{v_1, v_2, \ldots, v_n\}$ and arc set $E(T) = \{(v_i, v_j) : 1 \leq i < j \leq n\}$. We claim that $T$ is transitive. Let $(v_i, v_j)$ and $(v_j, v_k)$ be arcs of $T$. Then $i < j < k$. Since $i < k$, $(v_i, v_k)$ is an arc of $T$ and so $T$ is transitive. For $1 \leq i \leq n$, $\text{od} \ v_i = n - i$. Therefore, a score sequence of $T$ is $\pi : 0, 1, \ldots, n - 1$.

Next, we show that if $T$ is a transitive tournament of order $n$, then $0, 1, \ldots, n - 1$ is a score sequence of $T$. This is equivalent to showing that every two vertices of $T$ have distinct scores. Let $u$ and $w$ be two vertices of $T$. Assume, without loss of generality, that $(u, w)$ is an arc of $T$. Let $W$ be the set of vertices of $T$ to which $w$ is adjacent. Therefore, $\text{od} \ w = |W|$. For each $x \in W$, $(w, x)$ is an arc of $T$. Since $T$ is transitive, $(u, x)$ is also an arc of $T$. However then, $\text{od} \ u \geq |W| + 1$ and so $\text{od} \ u \neq \text{od} \ w$.

The proof of Theorem 7.13 shows that the structure of a transitive tournament is uniquely determined.

**Corollary 7.14** For every positive integer $n$, there is exactly one transitive tournament of order $n$.

Combining this corollary with Theorem 7.12, we arrive at yet another corollary.

**Corollary 7.15** For every positive integer $n$, there is exactly one acyclic tournament of order $n$. 

7.4. TOURNAMENTS

Although there is only one transitive tournament of each order \( n \), in a certain sense, which we now describe, every tournament has the structure of a transitive tournament. Let \( T \) be a tournament. We define a relation on \( V(T) \) by \( u \) is related to \( v \) if there is both a \( u \rightarrow v \) path and a \( v \rightarrow u \) path in \( T \). This relation is an equivalence relation and, as such, this relation partitions \( V(T) \) into equivalence classes \( V_1, V_2, \ldots, V_k \) (\( k \geq 1 \)). Let \( S_i = T[V_i] \) for \( i = 1, 2, \ldots, k \). Then each subdigraph \( S_i \) is a strong tournament and, indeed, is maximal with respect to the property of being strong. The subdigraphs \( S_1, S_2, \ldots, S_k \) are called the strong components of \( T \). So the vertex sets of the strong components of \( T \) produce a partition of \( V(T) \).

Let \( T \) be a tournament with strong components \( S_1, S_2, \ldots, S_k \), and let \( \tilde{T} \) denote that digraph whose vertices \( u_1, u_2, \ldots, u_k \) are in one-to-one correspondence with these strong components (where \( u_i \) corresponds to \( S_i, i = 1, 2, \ldots, k \)) such that \( (u_i, u_j) \) is an arc of \( \tilde{T}, i \neq j \), if and only if some vertex of \( S_i \) is adjacent to some vertex of \( S_j \). If \( (u_i, u_j) \) is an arc of \( \tilde{T} \), then because \( S_i \) and \( S_j \) are distinct strong components of \( T \), it follows that every vertex of \( S_i \) is adjacent to every vertex of \( S_j \). Hence \( \tilde{T} \) is obtained by identifying the vertices of \( S_i \) for \( i = 1, 2, \ldots, k \). A tournament \( T \) and its associated digraph \( \tilde{T} \) are shown in Figure 7.13.

![Figure 7.13: A tournament \( T \) and its associated transitive tournament \( \tilde{T} \)]

Observe that for the tournament \( T \) of Figure 7.13, \( \tilde{T} \) is itself a tournament, indeed a transitive tournament. That this always occurs follows from Theorem 7.16. (See Exercise 37.)

**Theorem 7.16** If \( T \) is a tournament with exactly \( k \) strong components, then \( \tilde{T} \) is the transitive tournament of order \( k \).

Since for every tournament \( T \), the tournament \( \tilde{T} \) is transitive, it follows that if \( T \) is a tournament that is not strong, then \( V(T) \) can be partitioned as \( \{V_1, V_2, \ldots, V_k\} \) (\( k \geq 2 \)) such that \( T[V_i] \) is a strong tournament for each \( i \), and if \( v_i \in V_i \) and \( v_j \in V_j \), where \( i < j \), then \( (v_i, v_j) \in E(T) \). This decomposition is often useful when studying the properties of tournaments that are not strong.

We already noted that there are four tournaments of order 4. Of course, one of these is transitive, which consists of four trivial strong components
There are two tournaments of order 4 containing two strong components $S_1, S_2, S_3, S_4$, where the vertex of $S_i$ is adjacent to the vertex of $S_j$ if and only if $i < j$. There are two tournaments of order 4 containing two strong components $S_1$ and $S_2$, depending on whether $S_1$ or $S_2$ is the strong component of order 3. (No strong component has order 2.) Since there are four tournaments of order 4, there is exactly one strong tournament of order 4. These tournaments are depicted in Figure 7.14. The arcs not drawn in the tournaments $T_1, T_2$ and $T_3$ that are not strong are all directed downward, as indicated by the double arrow.

We also stated that there are 12 tournaments of order 5. There are six tournaments of order 5 that are not strong, shown in Figure 7.15. Again all arcs that are not drawn are directed downward. Thus, there are six strong tournaments of order 5.

Score Sequences of Tournaments

Theorem 7.13 characterizes score sequences of transitive tournaments. We next investigate score sequences of tournaments in general. We begin with a theorem similar to Theorem 1.12.

**Theorem 7.17** A nondecreasing sequence $\pi : s_1, s_2, \ldots, s_n$ ($n \geq 2$) of non-negative integers is a score sequence of a tournament if and only if the sequence $\pi_1 : s_1, s_2, \ldots, s_{n-1}, s_{n+1}-1, \ldots, s_{n-1}-1$ is a score sequence of a tournament.
Assume that $\pi_1$ is a score sequence of a tournament. Then there exists a tournament $T_1$ of order $n-1$ having $\pi_1$ as a score sequence. Hence the vertices of $T_1$ can be labeled as $v_1, v_2, \ldots, v_{n-1}$ such that

$$\text{od } v_i = \begin{cases} s_i & \text{for } 1 \leq i \leq s_n \\ s_i - 1 & \text{for } i > s_n. \end{cases}$$

We construct a tournament $T$ by adding a vertex $v_n$ to $T_1$ where $v_n$ is adjacent to $v_i$ if $1 \leq i \leq s_n$ and $v_n$ is adjacent from $v_i$ otherwise. The tournament $T$ then has $\pi$ as a score sequence.

For the converse, we assume that $\pi$ is a score sequence. Hence there exist tournaments of order $n$ whose score sequence is $\pi$. Among all such tournaments, let $T$ be one such that $V(T) = \{v_1, v_2, \ldots, v_n\}$, $\text{od } v_i = s_i$ for $i = 1, 2, \ldots, n$ and the sum of the scores of the vertices adjacent from $v_n$ is minimum. We claim that $v_n$ is adjacent to vertices having scores $s_1, s_2, \ldots, s_{s_n}$. Assume, to the contrary, that $v_n$ is not adjacent to vertices having scores $s_1, s_2, \ldots, s_{s_n}$. Necessarily, then, there exist vertices $v_j$ and $v_k$ with $j < k$ and $s_j < s_k$ such that $v_n$ is adjacent to $v_k$ and $v_n$ is adjacent from $v_j$. Since the score of $v_k$ exceeds the score of $v_j$, there exists a vertex $v_t$ such that $v_k$ is adjacent to $v_t$, and $v_t$ is adjacent to $v_j$ (Figure 7.16(a)). Thus, a 4-cycle $C = (v_n, v_k, v_t, v_j, v_n)$ is produced. If we reverse the directions of the arcs of $C$, a tournament $T'$ is obtained also having $\pi$ as a score sequence (Figure 7.16(b)). However, in $T'$, the vertex $v_n$ is adjacent to $v_j$ rather than $v_k$. Hence the sum of the scores of the vertices adjacent from $v_n$ is smaller in $T'$ than in $T$, which is impossible. Thus, as claimed, $v_n$ is adjacent to vertices having scores $s_1, s_2, \ldots, s_{s_n}$. Then $T - v_n$ is a tournament having score sequence $\pi_1$. ■

Figure 7.16: A step in the proof of Theorem 7.17
As an illustration of Theorem 7.17, we consider the sequence
\[ \pi : 1, 2, 2, 3, 3, 4. \]
In this case, \( s_n \) (actually \( s_6 \)) has the value 4; thus, we delete the last term, repeat the first \( s_n = 4 \) terms, and subtract 1 from the remaining terms, obtaining
\[ \pi' : 1, 2, 2, 3. \]
Rearranging, we have
\[ \pi_1 : 1, 2, 2, 3. \]
Repeating this process twice more, we have
\[ \pi_2 : 1, 2, 2, 1 \]
\[ \pi_3 : 1, 1, 2. \]
\[ \pi_4 : 1, 1, 1. \]
The sequence \( \pi_3 \) is clearly a score sequence of a tournament. By Theorem 7.17, \( \pi_2 \) is as well, as are \( \pi_1 \) and \( \pi \). We can use this information to construct a tournament with score sequence \( \pi \).

The sociologist Hyman Garshin Landau [152] characterized those sequences of nonnegative integers that are score sequences of tournaments. The proof we present of his theorem is due to Carsten Thomassen [233].

**Theorem 7.18** A nondecreasing sequence \( \pi : s_1, s_2, \ldots, s_n \) of nonnegative integers is a score sequence of a tournament if and only if for each integer \( k \) with \( 1 \leq k \leq n \),
\[ \sum_{i=1}^{k} s_i \geq \binom{k}{2}, \tag{7.1} \]
with equality holding when \( k = n \).

**Proof.** Suppose first that \( \pi : s_1, s_2, \ldots, s_n \) is a score sequence of a tournament of order \( n \). Then there exists a tournament \( T \) with \( V(T) = \{v_1, v_2, \ldots, v_n\} \) such that \( \text{od}_T v_i = s_i \) for \( i = 1, 2, \ldots, n \). For an integer \( k \) with \( 1 \leq k \leq n \) and \( S = \{v_1, v_2, \ldots, v_k\} \), the subdigraph \( T_1 = T[S] \) induced by \( S \) is a tournament of order \( k \) and size \( \binom{k}{2} \). Since \( \text{od}_T v_i \geq \text{od}_{T_1} v_i \) for \( 1 \leq i \leq k \), it follows that
\[ \sum_{i=1}^{k} s_i = \sum_{i=1}^{k} \text{od}_T v_i \geq \sum_{i=1}^{k} \text{od}_{T_1} v_i = \binom{k}{2}. \]
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Figure 7.17: Construction of a tournament with a given score sequence

We now verify the converse. Suppose that the converse is false. Then among all counterexamples for which \( n \) is minimum, let \( \pi : s_1, s_2, \ldots, s_n \) be one for which \( s_1 \) is minimum. Suppose first that there exists an integer \( k \) with \( 1 \leq k \leq n - 1 \) such that

\[
\sum_{i=1}^{k} s_i = \binom{k}{2}.
\]  

(7.2)

Since \( k < n \), it follows that \( \pi_1 : s_1, s_2, \ldots, s_k \) is a score sequence of a tournament \( T_1 \) of order \( k \).

Let \( \tau : t_1, t_2, \ldots, t_{n-k} \) be the sequence, where \( t_i = s_{k+i} - k \) for \( i = 1, 2, \ldots, n-k \). Since

\[
\sum_{i=1}^{k+1} s_i \geq \binom{k+1}{2},
\]

it follows from (7.2) that

\[
s_{k+1} = \sum_{i=1}^{k+1} s_i - \sum_{i=1}^{k} s_i \geq \binom{k+1}{2} - \binom{k}{2} = k.
\]

Since \( \pi \) is a nondecreasing sequence,

\[
t_i = s_{k+i} - k \geq s_{k+1} - k \geq 0
\]

for \( i = 1, 2, \ldots, n-k \) and so \( \tau \) is a nondecreasing sequence of nonnegative integers. We now show that \( \tau \) satisfies (7.1).

For each integer \( r \) with \( 1 \leq r \leq n-k \), we have

\[
\sum_{i=1}^{r} t_i = \sum_{i=1}^{r} (s_{k+i} - k) = \sum_{i=1}^{r} s_{k+i} - rk = \sum_{i=1}^{r+k} s_i - \sum_{i=1}^{k} s_i - rk.
\]
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Since
\[ \sum_{i=1}^{r+k} s_i \geq \binom{r+k}{2} \]
and
\[ \sum_{i=1}^{k} s_i = \binom{k}{2}, \]
it follows that
\[ \sum_{i=1}^{r} t_i \geq \binom{r+k}{2} - \binom{k}{2} - rk = \binom{r}{2}, \]
with equality holding for \( r = n - k \). Thus, \( \tau \) satisfies (7.1). Since \( n - k < n \), there is a tournament \( T_2 \) of order \( n - k \) having score sequence \( \tau \).

Let \( T \) be the tournament with \( V(T) = V(T_1) \cup V(T_2) \) and
\[ E(T) = E(T_1) \cup E(T_2) \cup \{(u,v) : u \in V(T_2), v \in V(T_1)\}. \]
Then \( \pi \) is a score sequence for \( T \), contrary to our assumption. Consequently,
\[ \sum_{i=1}^{k} s_i > \binom{k}{2} \]
for \( k = 1, 2, \ldots, n - 1 \). In particular, \( s_1 > 0 \).

We now consider the sequence \( \pi' : s_1 - 1, s_2, s_3, \ldots, s_{n-1}, s_n + 1 \). Then \( \pi' \) is a nondecreasing sequence of nonnegative integers satisfying (7.1). By the minimality of \( s_1 \), there is a tournament \( T' \) of order \( n \) having score sequence \( \pi' \). Let \( x \) and \( y \) be vertices of \( T' \) such that \( \text{od}_{T'} x = s_n + 1 \) and \( \text{od}_{T'} y = s_1 - 1 \). Since \( \text{od}_{T'} x \geq \text{od}_{T'} y + 2 \), there is a vertex \( w \neq x, y \) such that \( (x, w) \in E(T') \) and \( (w, y) \in E(T') \). Thus, \( P = (x, w, y) \) is a path in \( T' \).

Let \( T \) be a tournament obtained from \( T' \) by reversing the directions of the arcs in \( P \). Then \( \pi \) is a score sequence for \( T \), producing a contradiction.

Frank Harary and Leo Moser [121] obtained a related characterization of sequences of nonnegative integers that are score sequences of strong tournaments (see Exercise 42).

**Theorem 7.19** A nondecreasing sequence \( \pi : s_1, s_2, \ldots, s_n \) of nonnegative integers is a score sequence of a strong tournament if and only if
\[ \sum_{i=1}^{k} s_i > \binom{k}{2} \]
for \( 1 \leq k \leq n - 1 \) and
\[ \sum_{i=1}^{n} s_i = \binom{n}{2}. \]
Furthermore, every tournament whose score sequence satisfies these conditions is strong.
7.5 Kings in Tournaments

While tournaments can be used to represent the results of round robin tournaments (especially among teams participating in a sports event), they can be used to model any collection of objects where in each pair of objects, one is preferred over the other in some manner. An example of this occurs in a flock of chickens. In a pair of chickens, one chicken will dominate the other. The dominant chicken in the pair asserts this dominance by pecking the other on its head and neck. (This is what led to the term *pecking order.*) It is rare when this dominance is transitive; that is, if the first chicken pecks a second chicken and the second pecks a third, it does not mean that the first necessarily pecks the third. The question then arises: Which chicken (or chickens) should be considered most dominant in the flock? Any such chicken is referred to as a *king chicken.* Landau [152] defined a chicken $K$ in a flock $F$ of chickens to be a *king* if for every chicken $C$ in $F$, either $K$ pecks $C$ or $K$ pecks a chicken that pecks $C$.

This situation can be modeled by a tournament and leads to a concept involving tournaments. A vertex $u$ in a tournament $T$ is a *king* in $T$ if for every vertex $w$ different from $u$, either $u \rightarrow w$ or there is a vertex $v$ such that $u \rightarrow v \rightarrow w$. Landau then proved the following.

**Theorem 7.20** Every tournament contains a king.

**Proof.** Let $T$ be a tournament and let $u$ be a vertex having maximum outdegree in $T$. We show that $u$ is a king. If this is not the case, then there is a vertex $w$ in $T$ for which $u$ is neither adjacent to $w$ nor adjacent to any vertex that is adjacent to $w$. Then $w$ is adjacent to every vertex to which $u$ is adjacent and adjacent to $u$ as well. Thus, $\text{od}(w) > \text{od}(u)$, a contradiction. ■

A vertex $u$ in a tournament of order $n$ is called an *emperor* if $\text{od}(u) = n - 1$. Since no vertex is adjacent to $u$, we have the following observation.

**Theorem 7.21** If a tournament $T$ has an emperor $u$, then $u$ is the unique king in $T$.

While it’s possible for a tournament to have exactly one king, it is not possible for a tournament to contain exactly two kings.

**Theorem 7.22** Every tournament containing no emperor contains at least three kings.

**Proof.** Let $T$ be a tournament containing no emperor and let $u$ be a vertex of maximum outdegree in $T$. By the proof of Theorem 7.20, it follows that $u$ is a king of $T$.

Among all vertices adjacent to $u$, let $v$ be one of maximum outdegree. We claim that $v$ is a king of $T$. Assume, to the contrary, that $v$ is not a king in $T$. Then there is a vertex $x$ of $T$ such that $v$ is neither adjacent to $x$ nor
adjacent to a vertex that is adjacent to $x$. Thus, $x$ is adjacent to both $u$ and $v$. Furthermore, $x$ is adjacent to every vertex to which $v$ is adjacent. However then, $\text{od } x > \text{od } v$, a contradiction. Thus, $v$ is a king of $T$.

Next, among the vertices adjacent to $v$, let $w$ be one of maximum outdegree. We claim that $w$ is also a king of $T$. Assume, to the contrary, that $w$ is not a king. Then there is a vertex $y$ of $T$ such that $w$ is neither adjacent to $y$ nor adjacent to a vertex that is adjacent to $y$. Thus, $y$ is adjacent to $v$ and $w$. In addition, $y$ is adjacent to every vertex to which $w$ is adjacent. However then, $\text{od } y > \text{od } w$, a contradiction. Hence $w$ is a king of $T$.

7.6 Hamiltonian Tournaments

The large number of arcs in a tournament often produce paths and cycles of varying lengths. Perhaps the most basic result of this type was a property of tournaments first observed by László Rédei [197] in 1934, resulting in the first theoretical result on tournaments. A path in a digraph $D$ containing every vertex of $D$ is a Hamiltonian path.

**Theorem 7.23** Every tournament contains a Hamiltonian path.

**Proof.** Let $T$ be a tournament of order $n$ and let $P = (v_1, v_2, \ldots, v_k)$ be a longest path in $T$. If $P$ is not a Hamiltonian path of $T$, then $1 \leq k < n$ and there is a vertex $v$ of $T$ not on $P$. Since $P$ is a longest path, $(v, v_1), (v_k, v) \notin E(T)$, and so $(v_1, v), (v, v_k) \in E(T)$. This implies that there is a largest integer $i$ ($1 \leq i < k$) such that $(v_i, v) \in E(T)$. So $(v, v_{i+1}) \in E(T)$ (see Figure 7.18). But then

$$(v_1, v_2, \ldots, v_i, v, v_{i+1}, \ldots, v_k)$$

is a path whose length exceeds that of $P$, producing a contradiction. }

![Figure 7.18: A step in the proof of Theorem 7.23](image)

A simple but useful consequence of Theorem 7.23 concerns transitive tournaments.

**Corollary 7.24** Every transitive tournament contains exactly one Hamiltonian path.
The preceding corollary is a special case of a result found independently by Rédei [197] and Tibor Széki [231], who showed that every tournament contains an odd number of Hamiltonian paths.

Figure 7.19 shows a tournament of order 5 consisting of three strong components $S_1$, $S_2$ and $S_3$, where $S_1$ and $S_3$ consist of a single vertex and $S_2$ is a 3-cycle. This tournament has three Hamiltonian paths, namely $P_1 = (u, v, w, x, y)$, $P_2 = (u, w, x, v, y)$ and $P_3 = (u, x, v, w, y)$.

![Tournament with three Hamiltonian paths](image)

While each transitive tournament contains exactly one Hamiltonian path, there are, not surprisingly, tournaments with many Hamiltonian paths. The next result, also due to Széki [231], establishes the existence of such tournaments and provides a lower bound on the number of Hamiltonian paths in them. The proof of this result, considered the first application of the probabilistic method in combinatorics, will be presented in Chapter 21 (see Theorem 21.3).

**Theorem 7.25** For each integer $n \geq 2$, there exists a tournament of order $n$ containing at least $n! / 2^{n-1}$ Hamiltonian paths.

While every tournament contains a Hamiltonian path, certainly not every tournament contains a Hamiltonian cycle. Indeed, by Theorem 7.12, every transitive tournament is acyclic. If a tournament $T$ contains a Hamiltonian cycle, then $T$ is strong by Theorem 7.4. Paul Camion [41] showed that the converse is true as well.

**Theorem 7.26** A nontrivial tournament $T$ is Hamiltonian if and only if $T$ is strong.

**Proof.** We have already seen that every Hamiltonian tournament is strong. For the converse, assume that $T$ is a nontrivial strong tournament. Thus, $T$ contains cycles. Let $C$ be a cycle of maximum length in $T$. If $C$ contains all of the vertices of $T$, then $C$ is a Hamiltonian cycle. So, assume that $C$ is not Hamiltonian, say

$$C = (v_1, v_2, \ldots, v_k, v_1),$$
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where \(3 \leq k < n\). If \(T\) contains a vertex \(v\) that is adjacent to some vertex of \(C\) and adjacent from some vertex of \(C\), then there must be a vertex \(v_i\) of \(C\) that is adjacent to \(v\) such that \(v_{i+1}\) is adjacent from \(v\). In this case,

\[
C' = (v_1, v_2, \ldots, v_i, v, v_{i+1}, \ldots, v_k, v_1)
\]

is a cycle whose length is greater than that of \(C\), producing a contradiction. Hence, every vertex of \(T\) that is not on \(C\) is either adjacent to every vertex of \(C\) or adjacent from every vertex of \(C\). Since \(T\) is strong, there must be vertices of each type.

Let \(U\) be the set of all vertices of \(T\) that are not on \(C\) and such that each vertex of \(U\) is adjacent from every vertex of \(C\), and let \(W\) be the set of those vertices of \(T\) that are not on \(C\) such that every vertex of \(W\) is adjacent to each vertex of \(C\) (see Figure 7.20). Then \(U \neq \emptyset\) and \(W \neq \emptyset\).

\[\text{Figure 7.20: A step in the proof of Theorem 7.26}\]

Since \(T\) is strong, there is a path from every vertex of \(C\) to every vertex of \(W\). Since no vertex of \(C\) is adjacent to any vertex of \(W\), there must be a vertex \(u \in U\) that is adjacent to a vertex \(w \in W\). However then,

\[
C'' = (v_1, v_2, \ldots, v_k, u, w, v_1)
\]

is a cycle whose length is greater than the length of \(C\), a contradiction.

If \(T\) is a Hamiltonian tournament, then, of course, every vertex of \(T\) lies on every Hamiltonian cycle of \(T\). Actually, every vertex of \(T\) lies on a triangle of \(T\) as well.

**Theorem 7.27** Every vertex in a nontrivial strong tournament belongs to a triangle.

**Proof.** Let \(v\) be a vertex in a nontrivial strong tournament \(T\). By Theorem 7.26, \(T\) is Hamiltonian. Thus, \(T\) contains a Hamiltonian cycle \((v = v_1, v_2, \ldots, v_n, v_1)\). Since \(v\) is adjacent to \(v_2\) and adjacent from \(v_n\), there is a vertex \(v_i\) with \(2 \leq i < n\) such that \((v, v_i)\) and \((v_{i+1}, v)\) are arcs of \(T\). Thus, \((v, v_i, v_{i+1}, v)\) is a triangle of \(T\) containing \(v\).

It is perhaps surprising that if a tournament is Hamiltonian, then it must possess significantly stronger properties. A digraph \(D\) of order \(n \geq 3\) is **pancyclic** if it contains a cycle of every possible length, that is, \(D\) contains a cycle
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of length $\ell$ for each $\ell = 3, 4, \ldots, n$ and is **vertex-pancyclic** if each vertex $v$ of $D$ lies on a cycle of every possible length. Frank Harary and Leo Moser [121] showed that every nontrivial strong tournament is pancyclic, while John W. Moon [173] went one step further by obtaining the following result. The proof given here is due to Carsten Thomassen.

**Theorem 7.28** Every nontrivial strong tournament is vertex-pancyclic.

**Proof.** Let $T$ be a strong tournament of order $n \geq 3$, and let $v_1$ be a vertex of $T$. We show that $v_1$ lies on an $\ell$-cycle for each $\ell = 3, 4, \ldots, n$. We proceed by induction on $\ell$.

Since $T$ is strong, it follows by Theorem 7.27 that $v_1$ lies on a 3-cycle. Assume that $v_1$ lies on an $\ell$-cycle $C = (v_1, v_2, \ldots, v_\ell, v_1)$, where $3 \leq \ell \leq n - 1$. We show that $v_1$ lies on an $(\ell + 1)$-cycle.

**Case 1.** There is a vertex $v$ not on $C$ that is adjacent to at least one vertex of $C$ and is adjacent from at least one vertex of $C$. This implies that for some $i$ ($1 \leq i \leq \ell$), both $(v_i, v)$ and $(v, v_{i+1})$ are arcs of $T$ (where all subscripts are expressed modulo $\ell$). Thus, $v_1$ lies on the $(\ell + 1)$-cycle $$(v_1, v_2, \ldots, v_i, v, v_{i+1}, \ldots, v_\ell, v_1).$$

**Case 2.** No vertex $v$ exists as in Case 1. Let $U$ denote the set of all vertices in $V(T) - V(C)$ that are adjacent from every vertex of $C$, and let $W$ be the set of all vertices in $V(T) - V(C)$ that are adjacent to every vertex of $C$. Then $U \cup W = V(T) - V(C)$. Since $T$ is strong, neither $U$ nor $W$ is empty and there is a vertex $u$ in $U$ and a vertex $w$ in $W$ such that $(u, w) \in E(T)$. Thus, $v_1$ lies on the $(\ell + 1)$-cycle $$(u, w, v_1, v_2, \ldots, v_{\ell-1}, u),$$

completing the proof.

**Corollary 7.29** Every nontrivial strong tournament is pancyclic.
Exercises for Chapter 7

Section 7.1. Introduction to Digraphs

1. (a) We saw in Theorem 1.14 that there exists no graph whose vertices have distinct degrees. Show that there exists a digraph of order 5 whose vertices have distinct outdegrees and distinct indegrees.

(b) Does there exist a digraph of order 5 whose vertices have distinct outdegrees but the same indegree?

2. Determine all digraphs of order 4 and size 4.

3. Show that for every positive integer \(k\), there exists a digraph of even order, half of whose vertices have outdegree \(a\) and half have outdegree \(b\) and \(a - b = k\).

4. If all vertices of a digraph \(D\) of order 5 have distinct outdegrees except for two vertices that have the same outdegree \(a\), then what are the possible values of \(a\)?

5. Prove or disprove: No digraph contains an odd number of vertices of odd outdegree or an odd number of vertices of odd indegree.

6. Prove or disprove: If \(D_1\) and \(D_2\) are two digraphs with \(V(D_1) = \{u_1, u_2, \ldots, u_n\}\) and \(V(D_2) = \{v_1, v_2, \ldots, v_n\}\) such that \(\text{id}_{D_1} u_i = \text{id}_{D_2} v_i\) and \(\text{od}_{D_1} u_i = \text{od}_{D_2} v_i\) for \(i = 1, 2, \ldots, n\), then \(D_1 \cong D_2\).

7. Prove that there exist regular tournaments of every odd order but there are no regular tournaments of even order.

8. Let \(T\) be a tournament with \(V(T) = \{v_1, v_2, \ldots, v_n\}\). We know that

\[
\sum_{i=1}^{n} \text{od} v_i = \sum_{i=1}^{n} \text{id} v_i = \left(\frac{n}{2}\right).
\]

(a) Prove that \(\sum_{i=1}^{n} (\text{od} v_i)^2 = \sum_{i=1}^{n} (\text{id} v_i)^2\).

(b) Prove or disprove: \(\sum_{i=1}^{n} (\text{od} v_i)^3 = \sum_{i=1}^{n} (\text{id} v_i)^3\).

9. The adjacency matrix \(A(D)\) of a digraph \(D\) with \(V(D) = \{v_1, v_2, \ldots, v_n\}\) is the \(n \times n\) matrix \([a_{ij}]\) defined by \(a_{ij} = 1\) if \((v_i, v_j) \in E(D)\) and \(a_{ij} = 0\) otherwise.

(a) What information do the row sums and column sums of the adjacency matrix of a digraph provide?

(b) Characterize matrices that are adjacency matrices of digraphs.

10. (a) Prove Theorem 7.2: If \(D\) is a digraph such that \(\text{od} v \geq k \geq 1\) for every vertex \(v\) of \(D\), then \(D\) contains a cycle of length at least \(k + 1\).
(b) Prove that if $D$ is a digraph such that $id_v \geq k \geq 1$ for every vertex $v$ of $D$, then $D$ contains a cycle of length at least $k + 1$.

11. Let $G$ be a connected graph of order $n \geq 3$. Prove that there is an orientation of $G$ containing no directed path of length 2 if and only if $G$ is bipartite.

12. Prove that for every two positive integers $a$ and $b$ with $a \leq b$, there exists a strong digraph $D$ with $\text{rad}(D) = a$ and $\text{diam}(D) = b$.

13. The center $\text{Cen}(D)$ of a strong digraph $D$ is the subdigraph induced by those vertices $v$ with $e(v) = \text{rad}(D)$. Prove that for every oriented graph $D_1$, there exists a strong oriented graph $D$ such that $\text{Cen}(D) = D_1$.

14. Prove that every digraph $D$ contains a set $S$ of vertices with the properties (1) no two vertices in $S$ are adjacent in $D$ and (2) for every vertex $v$ of $D$ not in $S$, there exists a vertex $u$ in $S$ such that $\vec{d}(u, v) \leq 2$.

Section 7.2. Strong Digraphs

15. Prove Theorem 7.3: Let $u$ and $v$ be two vertices in a digraph $D$. For every $u - v$ walk $W$ in $D$, there exists a $u - v$ path $P$ such that every arc of $P$ belongs to $W$.

16. Show that a digraph $D$ is strong if and only if its converse $\vec{D}$ is strong.

17. Let $G$ be a nontrivial connected graph without bridges.

(a) Show that for every edge $e$ of $G$ and for every orientation of $e$, there exists an orientation of the remaining edges of $G$ such that the resulting digraph is strong.

(b) Show that (a) need not be true if we begin with an orientation of two edges of $G$.

18. Let $G$ be a connected graph with cut-vertices. Show that an orientation $D$ of $G$ is strong if and only if the subdigraph of $D$ induced by the vertices of each block of $G$ is strong.

19. According to Theorem 7.5, a nontrivial graph $G$ has a strong orientation if and only if $G$ is connected and contains no bridges.

(a) Prove that if $G$ is a nontrivial connected graph with at most two bridges, then there exists an orientation $D$ of $G$ having the property that if $u$ and $v$ are any two vertices of $D$, there is either a $u - v$ path or a $v - u$ path.

(b) Show that the statement (a) is false if $G$ contains three bridges.
20. Let $D$ be a digraph of order $n \geq 2$. Prove that if $\text{od} v \geq (n - 1)/2$ and $\text{id} v \geq (n - 1)/2$ for every vertex $v$ of $D$, then $D$ is strong.

Section 7.3. Eulerian and Hamiltonian Digraphs

21. Prove Theorem 7.6: Let $D$ be a nontrivial connected digraph. Then $D$ is Eulerian if and only if $\text{od} v = \text{id} v$ for every vertex $v$ of $D$.

22. Prove that a graph $G$ has an Eulerian orientation if and only if $G$ is Eulerian.

23. Prove Theorem 7.7: Let $D$ be a nontrivial connected digraph. Then $D$ contains an Eulerian trail if and only if $D$ contains two vertices $u$ and $v$ such that $\text{od} u = \text{id} u + 1$ and $\text{id} v = \text{od} v + 1$, while $\text{od} w = \text{id} w$ for all other vertices $w$ of $D$. Furthermore, each Eulerian trail of $D$ begins at $u$ and ends at $v$.

24. Prove that if $D$ is a connected digraph containing two vertices $u$ and $v$ such that $\text{od} u = \text{id} u + k$ and $\text{id} v = \text{od} v + k$ for some positive integer $k$ and $\text{od} w = \text{id} w$ for all other vertices $w$ of $D$, then $D$ contains $k$ arc-disjoint $u - v$ paths.

25. Let $D$ be a digraph with an Eulerian trail. Then $D$ contains two vertices $u$ and $v$ such that $\text{od} u = \text{id} u + 1$ and $\text{id} v = \text{od} v + 1$, where $\text{od} w = \text{id} w$ for all other vertices $w$ of $D$.

(a) Let $T'$ be a $u - x$ trail in $D$ that cannot be extended to a longer trail. Must $x = v$?

(b) If $T$ is a $u - v$ trail in $D$, must $T$ be an Eulerian trail?

26. Prove that a nontrivial connected digraph $D$ is Eulerian if and only if $E(D)$ can be partitioned into subsets $E_i$, $1 \leq i \leq k$, where the subdigraph $D[E_i]$ induced by the set $E_i$ is a cycle for each $i$.

27. Prove that if $D$ is a connected digraph such that $\sum_{v \in V(D)} |\text{od} v - \text{id} v| = 2t$, where $t \geq 1$, then $E(D)$ can be partitioned into subsets $E_i$, $1 \leq i \leq t$, so that the subgraph $G[E_i]$ induced by $E_i$ is an open trail for each $i$.

28. Let $D$ be a connected digraph of order $n$ with $V(D) = \{v_1, v_2, \ldots, v_n\}$. Prove that if $\text{od} v_i \geq \text{id} v_i$ for $1 \leq i \leq n$, then $D$ is Eulerian.

29. A vertex $v$ in a digraph $D$ is said to be reachable from a vertex $u$ in $D$ if $D$ contains a $u - v$ path. Let $D$ be a digraph and for each vertex $u$ of $D$, let $R(u)$ be the set of vertices reachable from $u$ and let $r(u) = |R(u)|$. Since $u \in R(u)$ for every vertex $u$ of $D$, it follows that $r(u) \geq 1$. Prove that if $r(x) \neq r(y)$ for every two distinct vertices $x$ and $y$ of $D$, then $D$ contains a Hamiltonian path.
EXERCISES FOR CHAPTER 7

30. By Corollary 7.9, if $D$ is a nontrivial digraph of order $n$ such that $\text{od} u + \text{id} v \geq n$ when $(u, v) \notin E(D)$, then $D$ is Hamiltonian. Show that if $D$ is a nontrivial digraph of order $n$ such that $\text{od} u + \text{id} v \geq n - 1$ when $(u, v) \notin E(D)$, then $D$ is strong but may not be Hamiltonian.

31. Show for infinitely many positive integers $n$ that there exists a digraph $D$ of order $n$ such that $\text{od} v \geq (n - 1)/2$ and $\text{id} v \geq (n - 1)/2$ for every vertex $v$ of $D$ but $D$ is not Hamiltonian.

32. Corollary 7.10 states: If $D$ is a strong digraph of order $n$ such that $\deg v \geq n$ for every vertex $v$ of $D$, then $D$ is Hamiltonian. Show that if the digraph $D$ is not required to be strong, then $D$ need not be Hamiltonian.

Section 7.4. Tournaments

33. Give an example of two non-isomorphic strong tournaments of order 5.

34. How many tournaments of order 7 are there that are not strong?

35. Determine those positive integers $n$ for which there exist regular tournaments of order $n$.

36. Give an example of two non-isomorphic regular tournaments of the same order.

37. Prove Theorem 7.16. If $T$ is a tournament with exactly $k$ strong components, then $\overline{T}$ is the transitive tournament of order $k$.

38. (a) Show that if two vertices $u$ and $v$ have the same score in a tournament $T$, then $u$ and $v$ belong to the same strong component of $T$.

(b) Prove that every regular tournament is strong.

39. Which of the following sequences are score sequences of tournaments? For each sequence that is a score sequence, construct a tournament having the given sequence as a score sequence.

(a) 0, 1, 1, 4, 4
(b) 1, 1, 1, 4, 4
(c) 1, 3, 3, 3, 3, 5
(d) 2, 3, 3, 4, 4, 4, 5.

40. Show that if $\pi : s_1, s_2, \ldots, s_n$ is a score sequence of a tournament, then $\pi_1 : n - 1 - s_1, n - 1 - s_2, \ldots, n - 1 - s_n$ is a score sequence of a tournament.

41. What tournament $T$ of order $n$ has a score sequence $s_1, s_2, \ldots, s_n$ such that equality holds in (7.1) for every integer $k$ with $1 \leq k \leq n$?
42. Prove Theorem 7.19: A nondecreasing sequence \( \pi : s_1, s_2, \ldots, s_n \) of non-negative integers is a score sequence of a strong tournament if and only if 
\[
\sum_{i=1}^{k} s_i > \binom{k}{2} \quad \text{for} \quad 1 \leq k \leq n - 1 \quad \text{and} \quad \sum_{i=1}^{n} s_i = \binom{n}{2} .
\]
Furthermore, every tournament whose score sequence satisfies these conditions is strong.

43. We have seen that there is exactly one transitive tournament of each order. A tournament of order \( n \geq 3 \) is defined to be \textbf{circular} if whenever \((u, v)\) and \((v, w)\) are arcs of \( T \), then \((w, u)\) is an arc of \( T \).

(a) How many circular tournaments of order 3 are there?
(b) Show that in a tournament of order 3 or more, every vertex, with at most two exceptions, has positive outdegree and positive indegree.
(c) How many circular tournaments of order 4 or more are there?

44. For each positive integer \( k \), there exist round robin tournaments containing \( 2k \) teams with no ties permitted in which \( k \) of these teams win \( r \) games and the remaining \( k \) of these teams win \( s \) games for some \( r \) and \( s \) with \( r \neq s \). What is the minimum value of \( s \) for which this is possible?

45. Prove that if \( u \) and \( v \) are vertices of a tournament such that \( \vec{d}(u, v) = k \), then \( \text{id}_D u \geq k - 1 \).

46. For a tournament \( T \) of order \( n \), let 
\[
\Delta = \max\{\text{od}_T v : v \in V(T)\} \quad \text{and} \quad \delta = \min\{\text{od}_T v : v \in V(T)\} .
\]
Prove that if \( \Delta - \delta < \frac{n}{2} \), then \( T \) is strong.

47. Let \((u, v)\) be an arc of a tournament \( T \). Show that if \( \text{od}_T v > \text{od}_T u \), then \((u, v)\) lies on a triangle of \( T \).

48. Show that a tournament can contain three vertices of outdegree 1 but can never contain four vertices of outdegree 1.

49. Let \( T \) be a tournament of order \( n \geq 10 \). Suppose that \( T \) contains two vertices \( u \) and \( v \) such that when the arc joining \( u \) and \( v \) is removed, the resulting digraph \( D \) contains neither a \( u \rightarrow v \) path nor a \( v \rightarrow u \) path. Show that \( \text{od}_D u = \text{od}_D v \).

50. Let \( u \) and \( v \) be two vertices in a tournament \( T \). Prove that if \( \vec{d}(u, v) = k \geq 2 \), then \( T \) contains a cycle of length \( \ell \) for each integer \( \ell \) with \( 3 \leq \ell \leq k + 1 \).

51. Let \( u \) and \( v \) be two vertices in a tournament \( T \). Prove that if \( u \) and \( v \) do not lie on a common cycle, then \( \text{od}_T u \neq \text{od}_T v \).

52. Let \( T \) be a tournament with the property that every vertex of \( T \) belongs to a directed 3-cycle. Let \( u \) and \( v \) be distinct vertices of \( T \). Prove that if \( |\text{od}_T u - \text{od}_T v| \leq 1 \), then \( T \) contains both a directed \( u \rightarrow v \) path and a directed \( v \rightarrow u \) path.
Section 7.5. Kings in Tournaments

53. Show that every vertex in a nontrivial regular tournament is a king.

54. A tournament $T$ of order $n$ can only be regular if $n$ is odd and so $od_v = (n - 1)/2$ for every vertex $v$ of $T$. By Exercise 53, every vertex of $T$ is a king. Prove or disprove: There exists an even integer $n \geq 6$ such that for every tournament $T$ of order $n$ for which $od_v \geq (n - 2)/2$ for each $v \in V(T)$, every vertex of $T$ is a king.

55. Show that there exists a tournament of order 4 having exactly three kings.

56. Show that there exists a tournament of order 5 having exactly four kings.

57. Show that there is an infinite class of tournaments in which every vertex except one is a king.

58. A vertex $z$ in a nontrivial tournament is called a **serf** if for every vertex $x$ distinct from $z$, either $x$ is adjacent to $z$ or $x$ is adjacent to a vertex that is adjacent to $z$. Prove that every nontrivial tournament has at least one serf.

Section 7.6. Hamiltonian Tournaments

59. Prove that if $T$ is a tournament that is not transitive, then $T$ has at least three Hamiltonian paths.

60. (a) It has been mentioned that every tournament has an odd number of Hamiltonian paths. If $T$ is a tournament of order 5 that is not strong, then what is the maximum number of Hamiltonian paths that $T$ can have?

   (b) A tournament $T$ of order 9 has no strong components of order 5 or more and contains $k$ Hamiltonian paths. What are the possible values of $k$?

61. A tournament $T$ of order 10 contains $k$ Hamiltonian paths and consists of two strong components $S_1$ and $S_2$ of order 5. The strong component $S_1$ has $V(S_1) = \{v_1, v_2, \ldots, v_5\}$ and for $1 \leq i \leq 5$, $(v_i, v_j)$ is an arc of $S$ if $j = i + 1$ or $j = i + 2$ (addition modulo 5). Determine the number of Hamiltonian paths in $S_2$ in terms of $k$.

62. Prove or disprove: If every vertex of a tournament $T$ belongs to a cycle in $T$, then $T$ is strong.

63. (a) Prove or disprove: Every arc of a nontrivial strong tournament $T$ lies on a Hamiltonian cycle of $T$.

   (b) A digraph $D$ is **Hamiltonian-connected** if for every pair $u, v$ of vertices of $D$, there exists a Hamiltonian $u - v$ path. Prove or disprove: Every vertex-pancyclic tournament is Hamiltonian-connected.
64. Show that if a tournament \( T \) has an \( \ell \)-cycle, then \( T \) has an \( s \)-cycle for 
\[ s = 3, 4, \ldots, \ell. \]

65. A tournament \( T \) of order \( n \) contains a \( k \)-cycle \( C \) for some \( k \geq 4 \), no 
\( (k + 1) \)-cycle, a \( (k - 1) \)-cycle \( C' \) having no vertex on \( C \), a \( (k - 1) \)-cycle \( C'' \) 
having a vertex on \( C \) and \( k \) vertices lying on no cycle of \( T \). What is the 
minimum value of \( n \) in terms of \( k \).