CHAPTER 3  Structural element behaviour

In this chapter the behaviour of the structure that is part of a load path is examined in detail. The understanding that is obtained from this examination makes it clear how parts of structures resist the internal forces. It also gives guidance on the best shape or structural form for any particular part of the load path. The choice of overall structural form for any particular structure is one of the basic tasks of the structural designer but before the behaviour of whole structural forms can be understood, the behaviour of very simple structures must be clear. To do this it is helpful to think of structures being assemblies of elements.

3.1 Structural elements

For structural engineering convenience, elements are considered to be one dimensional, two dimensional or three dimensional. The basic element can be thought of as a rectangular block, with sides of dimensions A, B and C.

If the three dimensions are very approximately equal then such an element is a three-dimensional element. Examples of three-dimensional elements are rare in modern building structures but often occur in older buildings, such as wall buttresses or thick stone domes.
If one of the dimensions, say dimension $B$, is small compared with dimensions $A$ and $C$, then the element is a **two-dimensional element**.

Many parts of modern building structures are two-dimensional elements such as floor slabs, walls or shell roofs.

If two of the dimensions of the basic element, say $B$ and $C$, are small compared with dimension $A$, then the element is a **one-dimensional element**.

One-dimensional elements are used abundantly in nearly all buildings; examples are beams, bars, cables and columns. Using the concept of elements, structures can be conceived as assemblies of elements. Examples can be found both in traditional and modern structures.
Nowadays structures are usually conceived and designed as assemblies of structural elements. This means the structural behaviour can be quantified by considering the behaviour of each structural element in each load path.

### 3.2 Concepts of stress and stress distribution

For any structure, all the elements that make up each load path must be strong enough to resist the internal structural actions caused by the loads. This means detailed information is required about the structural behaviour of different materials and of the structural elements.
To obtain this knowledge a new concept has to be introduced, this is the concept of stress and the related idea of stress distribution. Stress is a word in common usage but for engineering it has a particular meaning and that is force per unit area. Stress distribution describes how the sizes of stresses vary from unit area to unit area.

To begin to understand these ideas it is helpful to look at the slice of a column shown in Fig. 2.2.

![Figure 2.2](image)

Suppose the cross-section of the slice is gridded into squares of the same size (unit squares), then a force can be attached to each square. If the slice is divided into 25 unit squares, so the force shown in Fig. 3.11 is divided into 25 forces per unit area, $f_1$ to $f_{25}$.

![Figure 3.11](image)

For equilibrium, the numerical sum of the sizes of the twenty-five forces per unit area must equal the total force on the cross-section. So far there is no requirement that any of the forces per unit area, $f_1$ to $f_{25}$, are numerically equal. Fig. 3.12 is redrawn as Fig. 3.13 showing a possible pattern of variation for $f_1$ to $f_{25}$.

![Figure 3.12](image)

![Figure 3.13](image)

The length of each force arrow indicates the size of the force in each unit square and it can be seen that these forces (stresses) vary in a pattern. Suppose, for clarity, just one strip of squares is drawn and the tops of the arrows are joined with a line.

![Figure 3.14](image)
As can be seen the resulting shape is a triangle, so along this strip there is a **triangular stress distribution**. Fig. 3.13 shows the stresses varying in both directions across the cross-section so the tops of all the arrows can be joined with lines as shown below.

![Fig. 3.15](image)

These lines show triangular shapes in one direction and rectangular ones in the other direction.

![Fig. 3.16](image)

It usual to simplify these diagrams of stress distribution by just drawing the outline along the edges.

![Fig. 3.17](image)

Notice that the stress distribution is drawn right across the section. The stress distribution shown in Fig. 3.17 is **triangular** in one direction and **uniform** in the other. Uniform, or constant, stress distribution means that the sizes of the stresses do not vary in that direction.

In general there is no restriction on how stresses vary across any cross-section of any structure, except that the sum of the stresses must be equal to the internal force acting at the section and the internal force acts at the **centre of gravity** of the stresses. This new concept of centre of gravity has been used, in a different way, for the see-saw (see Fig. 1.39). Suppose a see-saw has people of different weights all along its length, their weights are indicated by their sizes.
In this figure the see-saw is divided into ten equal spaces and, to balance, pairs of people of equal weight sit at equal distances from the balance point.

Two such pairs are shown. Provided the equal pairs sit at equal distances from the balance point, the order of the pairs does not matter. Fig. 3.20 shows two different seating arrangements both of which balance. That is because the centre of gravity of the ten people is at the balance point of the see-saw.

As the seated height of the people relates directly to their weight, these diagrams are conceptually similar to the diagrams of stress distributions. For instance the left-hand seating arrangement shown in Fig. 3.20 relates to a triangular stress distribution.

If the seating arrangement were altered to have all the heaviest people at one end, Fig. 3.22, then the balance point would have to be moved.
The new balance point will be nearer the end of the heavier people than the lighter. This is because the centre of gravity of this new seating arrangement is no longer at the centre of the see-saw. In the same way the internal force at any point of any structural element must act at the centre of gravity of the stress distribution. Where stresses vary in two directions across the section, the centre of gravity will also vary in two directions.

This new concept of stress allows checks to be made along each load path to ensure that it is strong enough to resist the internal forces caused by the loads. This is checked by making sure that the stresses in the structural elements that are in the load path are less than the maximum allowable (or usable) stress allowed for the structural material being used. In other words the structure must not be over-stressed. How a maximum allowable stress is decided is far from straightforward and is discussed later.

Using the concepts of load path, structural action and maximum stress, the main parts of the process of structural design can be outlined. Once the reason for the existence for the structure has been identified – building, water tank, bridge – then the process is used as follows:

**Step 1** Choose a structural form and material or materials.

**Step 2** Identify the loads that the structure has to carry.

**Step 3** Find the structural actions in the load path for each loadcase.

**Step 4** Check that each load path is not over-stressed.

The details of the process of structural design are examined later but now the concept of stress has been explained, the main steps of the process can be stated. This gives the basic framework that allows the overall behaviour of structures to be understood or designed.

The main point about the size of stresses is that they can be varied without altering the force. Carrying out step 4 of the design process may indicate that some part of a load path is over-stressed. If this is the case then it may be convenient to alter the structure locally, by altering the geometry, so that the stress is reduced below the maximum stress that is allowed.

This idea is used widely in everyday life; stresses are increased or reduced purposely. For example, the weight of a person may be constant, but the stress under the person’s feet will vary with the area of the shoe in contact with the ground. This variation may have good or bad effects. Fig. 3.23 shows three types of shoes, normal shoes, high heeled shoes and snow shoes.

![Fig. 3.23](image)

Normal shoes cause normal stresses and can be used on surfaces that can resist these stresses. High heeled shoes, as they provide a much smaller area to carry the same
weight, cause higher stresses under the shoe, particularly under the heel. With very slender (stiletto) heels the stresses can be high enough to permanently damage some types of normal floor surfaces. Where stresses must be kept low, for walking on snow for instance, the area under the foot must be increased. This is why snow shoes prevent people from sinking into snow.

The meaning of ‘comfortable’ shoes, beds and chairs etc. is partly based on limiting the stresses on the human body to ‘comfortable’ ones. Padded chairs with large seat areas are more comfortable than hard chairs with small seat areas.

![Fig. 3.24](image1)

![Fig. 3.25](image2)

The idea of deliberately altering stress sizes by geometric methods is also widely used in many other objects used by humans. For example drawing pins are provided with large heads, to allow comfortable stresses on the thumb, and pointed shafts to cause high stresses under the point. The point stress is so high that the base surface fails and allows the drawing pin to be driven in.

The important idea is that for equilibrium, the force on the head must equal the force on the point, but the stresses vary. The stresses are varied by changing the geometry (of the drawing pin). The provision of handles, points, sharp edges and wide shoulder straps are all familiar devices for deliberately raising or lowering stresses.

Returning to engineering structures, the task is to provide a structure that will carry the prescribed loads down the load path with ‘comfortable’ stresses everywhere. Depending on the material used, the size of the comfortable stress will vary. For instance as steel is stronger than timber, the allowable (comfortable) stress for steel is higher than for timber. So, in a general sort of way, timber structures will have larger structural elements than steel structures if they are to carry the same load.

As it is usually impractical to arrive at satisfactory structures by guesswork or by testing whole structures, the modern approach is to calculate the size of the stresses on each loadpath and to check that all the stresses are within set limits. For this to be a practical proposition rather than a research project, many simplifying assumptions have to be made. These assumptions allow what is usually known as Engineer’s theory to be used for stress calculations. Some of these assumptions are concerned with the nature of the material from which the structure is made. These are:

- **The material is isotropic.** This means that the mechanical behaviour of the material is the same in all directions.
• **The material is linear elastic.** An elastic material is one which after deforming under load returns to exactly the same state when the load is removed. If an elastic material deforms as an exact proportion of the load then it is linear elastic. This is discussed further in Chapter 6.

There are also assumptions about the geometry of the structure. These are:

• **The structure is homogeneous.** This means that there are no cracks, splits or holes or other discontinuities in the structure.

• **The deflections of the loaded structure are small.** This means that using the shape of the unloaded structure for calculations to determine structural behaviour will not lead to any significant errors. This does not apply to very flexible structures; think for example of a washing line.

• **Plane sections remain plane.** This rather cryptic statement means that certain parts of a structure that are flat before loading are still flat after loading. This is explained more fully in this chapter.

The Engineer’s theory is used for most structural design because it leads to simple stress distributions in structural elements when they are subjected to internal forces of bending moments, shear and axial forces.

### 3.3 Axial stresses

An axially loaded structural element has axial internal forces and these cause axial stresses across the element (see Figs. 3.11 and 12). Using the Engineer’s theory this leads to a very simple stress distribution.

![Fig. 3.26](image)

This plane cross-section remains plane after the column is axially loaded. This is visually implied in Fig. 2.2 where a slice of a loaded column is shown. What the plane sections remain plane assumption means in this case is that the flat faces of the unloaded slice are flat after the slice is loaded.

![Fig. 3.27](image)

This assumption gives a very simple stress distribution for an axially loaded column. Because the loaded faces remain flat, all parts of the column cross-section deflect by the same amount.
Because the deflections are equal over the cross-section, the stress (load/unit area) is the same everywhere, in other words there is a uniform stress distribution.

The uniform stress over the cross-section of an axially loaded column gives a very simple relationship between force and stress and this is:

- **Axial stress** = Axial force divided by the cross-sectional area

This means that for a given force, the size of the stress can be varied by increasing or decreasing the cross-sectional area of the column.

The assumption that plane sections remain plane also gives guidance as to when a structural element should be regarded as one, two or three dimensional (see Figs. 3.1, 3.3 and 3.5).

The assumption implies that the whole cross-section is equally stressed. Fig. 3.30 shows three columns each subjected to a local load. For the widest column it does not seem reasonable to assume that the whole cross-section is equally stressed or even that the whole cross-section is stressed.
This figure shows the stressed part of the three columns. Very approximately the stress ‘spreads out’ at about 60°. This means that for the widest column, plane sections do not remain plane. The faces of the loaded slice of the widest column do not remain plane.

![Fig. 3.32](image1)

From an engineering point of view this gives guidance as to whether structural elements are one, two or three dimensional. Where simple stress distributions are reasonable then elements can be regarded as one dimensional, but where the stress distributions are no longer simple, the elements are two or three dimensional. In Fig. 3.32 the widest column has to be regarded as a two dimensional element. This effect can be seen by pulling on progressively wider and wider sheets of paper. The stressed part of the paper will become taut; the unstressed areas will remain floppy.

![Fig. 3.33](image2)

### 3.4 Bending stresses

Where parts of the load path are spanning elements, beams and slabs, the elements will have internal bending forces (moments). The top and bottom surfaces of these elements become curved; however, plane cross-sections still remain plane.

![Fig. 3.34](image3)

Again looking at unloaded and loaded slices (see page 39), the plane sections can be identified.
By viewing the slice from the side it can be seen that the plane sections rotate.

When the slice is bent by an internal bending moment, AB is squashed, EF is stretched and CD remains the same length. Because the cross-sections remain plane, the amount each part of the slice is squashed or stretched varies directly with the distance it is from CD.

As the structural material is linear elastic, the force is directly proportional to the deflection, so the maximum compression is at AB and the amount of compression decreases constantly from AB to CD. Similarly the maximum tension is at EF and the tension decreases constantly from EF to CD. The maximum compression is at the top of the slice and the maximum tension is at the bottom of the slice and at CD, the change point, there is neither compression nor tension. Using this information a stress distribution diagram can be drawn for the side view of the slice.
If it is also assumed that these stresses that are caused by an internal bending moment do not vary across the beam, a three-dimensional diagram of the stress distribution of the compressive and tensile stresses can be drawn.

This stress distribution, which is based on the assumptions of linear elasticity and plane sections remaining plane, is widely (but not exclusively) used in structural engineering. It can be viewed as being in two parts, a triangular distribution of compressive stress and a triangular distribution of tensile stress. Figs. 3.21 and 3.22 explain how a stress distribution balances a force at the centre of gravity of the stress distribution. The two parts of the stress distribution give a new concept which is the moment as a pair of forces. In Figs. 2.9 to 2.12 a bending moment is shown as a rotating force. Now the bending moment acting on a slice of a beam can be thought of in three alternative ways – as a rotating force, as a double triangular distribution of compressive and tensile stresses, or as a pair of forces.

This figure illustrates a key concept in the understanding of structural behaviour and applies, often disguised, to almost all structures. These three alternative views are logically connected by the various concepts that have been introduced; really three steps have been made.

**Step 1** Connects the idea of a bending moment in a beam with plane sections remaining plane and the sides of the slice of a beam rotating.

**Step 2** Connects the deflection of the slice caused by the rotation of the sides to ideas of linear elasticity and stress distribution.
Step 3  This uses the idea that if a force causes a stress distribution, then where there is a stress distribution there must be a force. And this force must act at the centre of gravity of the stress distribution.

Fig. 3.43

In this figure the distance between the push forces, which are the effect of the compressive stresses, and the pull forces, which are the effect of the tensile stresses, is called the lever arm. Remembering that any moment is a force times a distance, the push and pull forces ‘give back’ the bending moment. Here, rather confusingly, the force can be the push or the pull force and the distance is the lever arm.

Fig. 3.44

The push and pull forces and the lever arm show how by altering the local geometry of the beam, the size of the stresses can be altered for any bending moment. From Fig. 3.44 two statements can be made about the sizes of the forces from the requirements of equilibrium. Firstly the forces on each face must be in horizontal equilibrium.

Fig. 3.45

So the first statement is:

- the size of the push force must equal the size of the pull force.

Secondly, from moment equilibrium, the bending moment must equal a force times the lever arm. So the second statement is:

- the size of the push force times the lever arm must equal the size of the pull force times the lever arm must equal the size of the bending moment.

The size of the bending moment is ‘fixed’ by the position of the element in the load path and the size of the loads the load path has to carry. So, from the second statement, if the lever arm is made bigger, the push (or pull) force is smaller and vice versa.
Because of statement two, the size of \textbf{PUSH FORCE 1} times \textbf{LEVER ARM 1} must equal \textbf{PUSH FORCE 2} times \textbf{LEVER ARM 2}. As \textbf{LEVER ARM 1} is bigger than \textbf{LEVER ARM 2} then \textbf{PUSH FORCE 1} will be smaller than \textbf{PUSH FORCE 2}. The relationship between the size of stresses and forces is dependent, for any force, on the area and the shape of the distribution. For bending stresses the distribution shown in \textbf{Fig. 3.39} has been used.

All the compressive stresses (force per unit area) on the upper part of the beam must add up to the push force, and all the tensile stresses on the lower part of the beam must add up to the pull force. By varying the \textbf{depth} and therefore the lever arm, the size of the push and pull forces can be altered, which means the sizes of the stresses can be altered. This is only true if the width of the beam is not altered. The size of the stresses can also be altered by varying the \textbf{width} because this alters the area. Or the size of the stresses can be altered by varying both the depth and the width.

Unlike axially loaded elements that are equally stressed over the whole cross-section (\textbf{Fig. 3.29}), beams bent by moments have varying stresses that are at a maximum at the top and bottom. As all structural materials have a maximum usable stress, rectangular solid beams like those shown in \textbf{Fig. 3.48} are \textbf{under-stressed} except for the top and bottom faces.

It is one ambition of structural design to try and stress all parts of a structure to the \textbf{maximum usable stress} of the structural material being used. In this way no
structural material is wasted. This is a sensible ambition provided it does not lead to geometrically complex structures that are expensive to build.

Not only can material be wasted within the depth of a beam but it can also be wasted along its length. Suppose a beam of constant depth and rectangular cross-section is used to carry a load over a simple span. The size of the bending moment will vary along the length of the beam.

![Fig. 3.50](image)

For this simple structure, the maximum stress only occurs at one place where the bending moment is at its maximum. Almost the whole of the beam has bending stresses less than the maximum. This contrasts sharply with a column with end loads. Here the whole of the cross-section and the whole of the length of the column can be at maximum stress and so none of the structural material is wasted.

![Fig. 3.51](image)

To try and make beams more stress effective, non-rectangular shapes have been developed. Although there is some visual evidence that ancient builders were aware of the effect of cross-sectional shapes on the bending performance of beams, the pioneers of modern engineering in the early 19th century took some time to evolve efficient shapes. As the maximum stresses for bending are at the top and bottom of a beam, more efficient beam sections have more structural material here. These efficient sections are I, channel or box sections.

![Fig. 3.52](image)

The exact details of these shapes depend on the structural material used, as the methods of construction are different. Furthermore where bending efficiency is not of paramount importance or for a variety of other reasons, such as cost and speed of construction, other shapes such as tubes, rods and angles may be used.

To understand why the shapes shown in Fig. 3.52 are bending efficient it is helpful to compare an I shaped section with a + shaped section. Both have the same depth and the same cross-sectional area.
As plane sections are assumed to remain plane, and both sections are assumed to have the same maximum usable stress, the side view of the stress distribution is the same as Fig. 3.38 for both the sections.

However, if the three-dimensional stress diagrams are drawn, similar to Fig. 3.39, dramatic differences appear.

The I section has large areas of the cross-section with stresses near to the maximum, but the + section has large areas with stresses near to zero. This means that the push and pull forces of Fig. 3.43 are much bigger for the I section than for the + section. Also the positions of the centres of gravity of these stresses are different and this gives the I section a larger lever arm than the + section.
The maximum bending moment a beam with any particular cross-section can carry is given by the second statement stated on page 70, and this moment is:

- **the push force, with the maximum usable stress, times the lever arm**

which is the same as:

- **the pull force, with the maximum usable stress, times the lever arm**

Both the lever arm and the push force (or pull force) are greater for the I section than for the + section. Because of this, if beams have the same depth, the same cross-sectional area and the same maximum usable stress, then those with I sections will be able to resist larger bending moments than those with + sections. Although I beams, as they are called, can be made from timber or reinforced concrete they are readily made from steel. Due to the bending efficiency of I beams they are very widely used in steel construction as any visit to a steel construction site will show.

By adopting more efficient cross sections, more structural material is used at, or near to, the maximum usable stress. But as the size of the bending moment usually varies along a beam, higher stresses can be achieved away from the position of maximum bending moment by reducing the width or the depth of the beam.

![Bending Moment Diagram](image)

**Fig. 3.57**

By reducing the depth the lever arm is reduced, so that the push and pull forces are higher for the smaller bending moment. Reducing the width has the effect of reducing the stress area for the push and pull forces and so increasing the stresses.

Building a beam with an I section and varying the depth or width to keep the bending stresses high where the size of the bending moment reduces, is an efficient use of structural material compared with using a solid rectangular section of constant depth and width. Whether it is worthwhile using the more complex bending efficient beam depends on cost, both of the material and cost of construction. It is common to see beams of varying depth used for road bridges but unusual in building structures. It is also common to see steel I beams, but timber structures, particularly in houses, nearly always use timber beams of rectangular cross-section of constant width and depth.

As with columns (see section 3.3), the assumption that plane sections remain plane is not always valid. There are two situations where it may not apply. The first is when the span of the beam is not more than about five times the depth of the beam. As the plane sections are no longer plane, the bending stress distribution is not the one shown in **Figs. 3.38 and 3.39**.

![Bending Stress Distribution](image)

**Fig. 3.58 – bending stress in a deep beam**
These beams, called **deep beams**, cannot be regarded as one-dimensional elements but are two-dimensional elements (see figure 3.3).

If a beam is not deep but is made from an I or similar section, again plane sections may not remain plane. If the widths of the top and bottom parts of the section are increased eventually they will become **too wide** and not all of the section will be stressed by the bending moment.

![Fig. 3.59 Normal I Beam vs Too Wide I Beam](image)

For a normal I beam the bending stresses are assumed to be constant across the top and bottom parts, but for wide beams only part of the beam may be stressed and the stress is not constant across the beam.

![Fig. 3.60 Normal Beam vs Wide Beam](image)

The effect that causes this varying stress across wide beams is called **shear lag** and the part of the beam that is stressed is often called the **effective width**.

### 3.5 Shear stresses

Axial forces cause axial stresses (see Fig. 3.29) and bending moments cause bending stresses (see Fig. 3.39), so it is not unreasonable to expect shear forces to cause **shear stresses**. Shear stresses resist vertical loads so it is to be expected that shear stresses act vertically. For the vertical shear force acting on the face of the slice of the beam, ideas similar to those shown in Fig. 3.12 can be used. Here, unlike the column, the shear stresses (force per unit area) act in line with the face of the slice.

![Fig. 3.61 Shear Stress](image)
Unfortunately the distribution of shear stress cannot be deduced from the straightforward assumptions that were used for axial and bending stress. At the top and the bottom the shear stress must be zero otherwise there would be vertical shear stresses on the surface of the beam, which is impossible. So what shape is the distribution of shear stress? Mathematical analysis shows that for a rectangular beam the shear stress distribution has a curved shape, accurately described as **parabolic shear stress distribution**. The maximum is at the middle of the beam and it is zero at the top and bottom and is constant across the width of the beam.

It is common to assume for practical structural engineering design that the shear stress distribution is rectangular rather than curved. This means that this shear stress is 50% less than the maximum shear stress and that there are vertical shear stresses at the top and bottom faces. In spite of these inaccuracies this assumption is thought to be worthwhile as it simplifies the numerical calculation of shear stresses.

With this assumption there is a similar relationship between shear force and shear stress as that used for the axial forces and stresses (see pages 65-66) and this is:

- Shear stress = Shear force divided by the shear area

The term shear area is introduced because for non-rectangular cross-sectional shapes the vertical area is used rather than the total area.

This figure shows typical shear areas for a few common structural cross-sections and illustrates the general idea of vertical shear area.
3.6 Torsional stresses

When a one-dimensional element is twisted by torsional moments (see section 2.8) the internal forces in the element cause torsional stresses. To see what is happening, cut a slice from a circular bar that is being twisted by torsional moments.

If the circular cross-section is divided into small areas by radial and circumferential lines then each area has a force in the tangential direction to the circumference. Compare this with the diagram for shear stresses — Fig. 3.61.

At the centre these tangential forces are zero and it is assumed that they increase linearly towards the outside of the bar. Compare this with the variation of bending stresses shown in Fig. 3.38.

If a circular tube, with a wall thickness that is ‘small’ compared with its diameter, is twisted by torsional moments, then it could be considered reasonable that the tangential stresses are constant across the wall of thickness $t$.

In this case the relationship between the torsional moment, $M_T$, and the tangential torsional stress, $f_T$, is particularly simple;
Torsional moment = Circumference at wall centre x wall thickness x torsional stress

or

\[ M_T = 2\pi r \times t \times f_T \]

The tangential torsional stresses in circular elements can be thought of as a number of loops or circles of constant stress, with the stress in each circle being proportional to the distance from the centre. The circular tube is then a special case of just one circle of constant stress.

As far as torsional stresses are concerned, circular rods and tubes are special cases because their cross-section is symmetrical about all radial axes. The faces of slices cut from these elements are flat (plane) before being twisted and remain flat after the element has been twisted – this is a special case and is not true for sections of other shapes. This should be compared with the idea of plane cross-sections shown in Fig. 3.35 for the case of a bent beam. If a rod with an elliptical cross-section is twisted, the face of a slice that is flat before the rod is twisted by a torsional moment will not remain flat. In this case the face will deform into a curved surface technically called a hyperbolic paraboloid. In other words the cross-section warps and the element deforms unequally in the longitudinal direction.

In the case of elliptical rods, the torsional stresses can still be thought of as a series of loops but in this case these loops are elliptical.

In the case of a rectangular bar, the loops are not all of the same shape and the cross-section warps in a more complex manner.
Where cross-sections of an element are made up of a number of rectangular elements which do not form any type of tube – I beams and channels for example – they are called open sections (see Fig. 3.52). For these sections the torsional stresses are loops round the whole section and the cross-section warps.

The torsional behaviour just described has two important features that are:

- the torsional stresses form ‘loops’ within the section
- in general a plane cross-section warps when the element is twisted

This type of torsional behaviour is often called Uniform Torsion or St. Venant’s Torsion (after the French mathematician Adhémar JCB de Saint-Venant (1797-1886) who presented the mathematical theory of torsion in 1853). Torsion is looked at again, from a different point of view, in the next chapter.

### 3.7 Curved elements

When structural elements are curved vertically or horizontally the internal forces, axial forces, bending moments and torsional moments cannot always be considered as separate internal forces as they can be interconnected. To see how this happens, two cantilevers, one curved vertically, one curved horizontally and each loaded at the end with a point load \( P \), are considered.
Looking at the horizontal cantilever, first consider an ‘L’ shaped cantilever with straight elements $AB$ and $BC$. Using definition that a bending moment is a distance times a force – see page 27 – the bending moment will increase linearly along $AB$ and be constant along $BC$. There will be no twisting effect in element $AB$ but a constant one in element $BC$. The bending moment and torsional moment diagrams are drawn in the next figure. A constant shear force of $P$ also exists, but the diagram is not drawn for this structure.

But with the curved cantilever, the bending and torsional moments are present throughout the element and vary continuously as trigonometrical functions of the angle $\theta$.

A similar approach is used for the vertical cantilever, firstly an ‘L’ shaped cantilever is considered. In this case only an axial force exists in element $AB$ whereas bending moments and shear forces exist in element $BC$.

Like the horizontal curved cantilever, all the forces in the vertical cantilever are present throughout the element again varying as trigonometrical functions of the angle $\theta$. 
If a curved member is bent to a tight radius, the distribution of stresses across the section, even using the assumption of the **Engineer's theory** (see page 64-65), will not be linear, however the linear distribution is accurate enough for most situations. Curved elements are not considered further in this book – the interested reader should consult the specialised literature. This brief section has been included to show how a structure with curved elements has a more complex and interactive behaviour than those structures that only have straight elements.

### 3.8 Combined stresses

When a one-dimensional element is part of a load path it will have internal forces and these may be axial forces, bending moments or shear forces. These internal forces can be thought of as distributions of axial stress, bending stress and shear stress. With the simplifying assumptions that have been made these stresses have very simple stress distributions.

For axial forces and axial stresses:

![Axial Force and Uniform Stress](Fig. 3.79)

for bending moments and bending stresses:

![Bending Moment and Stress](Fig. 3.80)

and for shear forces and shear stresses:

![Shear Force and Uniform Stress](Fig. 3.81)

**Figs. 2.45-2.47** shows how axial forces, bending moments and shear forces vary around a portal frame when it is loaded with a point load. Each part has an axial force, a bending moment and a shear force, causing distributions of axial stresses and bending stresses and shear stresses. Can these stresses be combined to give the total stress distribution? One way is to combine the stresses on the face of a slice; this type of combination is frequently used in engineering.
This way of combining stresses is relatively straightforward as it just adds stresses that are in the same direction on the face of the slice. Both axial stresses and bending stresses act at right angles to the face of the beam, which is along the beam, so they are combined by adding the stress distributions together.

![Diagram](image)

In this figure because the size of the axial compressive stress is bigger than the maximum tensile bending stress, the whole of the cross-section is in compression. The effect of combining the stresses gives a combined maximum stress and a combined minimum stress. The sizes of these stresses are:

- **Maximum stress** = Axial compressive stress plus Maximum compressive bending stress
- **Minimum stress** = Axial compressive stress minus Maximum tensile bending stress

Because the shear stress is at right angles to the face of the slice, it is not added to the axial and bending stresses but is kept separate. Depending on the relative sizes of the axial and bending stresses and whether the axial stress is tensile or compressive, the combined stress distribution is all tensile, tensile and compressive, or all compressive.

![Diagram](image)

The axial stress distribution can be thought of as an axial force acting at the centre of gravity of the axial stress distribution and the bending stress distribution can be thought of as a pair of push-pull forces acting at the centres of gravity of the tensile and compressive parts of the bending stress distribution.

![Diagram](image)

As the stress distributions have been combined to give one stress distribution, can the forces be combined to give one force? If so what is this force and where does it act? As Fig. 3.45 shows the push equals the pull so the combined force can only be the axial force. But this force must act at the centre of gravity of the combined stress distribution.

82 Chapter 3
And this centre of gravity is not at the centre of gravity of the axial stress.

The effect of the moment is to ‘move’ the axial force by a distance, \( e \), from the centre of gravity for uniform axial stress. This distance \( e \) is called the **eccentricity** and with this new concept many common engineering situations can be better understood.

Before combining the forces there was an axial force \( P \) and a bending moment \( M \). Now there is an axial force \( P \) that has ‘moved’ by a distance, the eccentricity \( e \). What has happened to \( M \), the bending moment? The bending moment still exists but now as \( P \) times \( e \). This is the by now familiar force \( P \), times distance \( e \).

This idea of the axial force acting at an eccentricity can be used for both internal forces and external forces. If a structural element has an internal axial force and a bending moment then this can be viewed as being the same as the axial force being applied at a point eccentric from the centre of gravity for uniform stress. Alternatively if an external axial load is applied to a structural element at a point eccentric from the centre of gravity for uniform axial stress then this can be viewed as being the same as applying an **axial load plus a moment**. This gives a very simple relationship between axial force, bending moment and eccentricity, which is

- **bending moment** = **axial force** times the **eccentricity**

or

- **eccentricity** = **bending moment** divided by the **axial force**

Suppose a beam is supported on a wall as in Fig. 1.52. Then, for the wall only to have uniform axial stress from the reaction of the beam, the beam must be supported exactly at the position of the centre of gravity for this uniform stress distribution.
This is usually impossible in any real structure unless very precise precautions are taken. This means the reaction from the beam that the wall is supporting, will be applied to the wall at an eccentricity. So the wall is loaded by an axial load plus a bending moment.

![Fig. 3.89](image)

In this figure the eccentricity is within the width of the wall but this will not always be the case. What happens at the base of a garden wall or any other free-standing wall, when the wind blows? The axial force is caused by the weight of the wall itself and the bending moment is caused by the wind blowing horizontally on the wall.

![Fig. 3.90](image)

Here the eccentricity could be of any size depending on the relative sizes of the axial force caused by the weight of the wall and the moment caused by the wind. The left diagram in **Fig. 3.91** show a cross-section with only compression stresses, whilst the right diagram shows compressive and tensile stresses. This means that the eccentricity is greater in the right diagram.

![Fig. 3.91](image)

For rectangular sections the eccentricity must be kept within the **middle third** of the cross-section if there is to be **no tensile stress**.

![Fig. 3.92](image)

This has very important consequences for structures made from structural materials such as masonry or mass concrete that cannot carry significant tensile stresses. For structures made from these materials, axial forces must be kept within the central part of the cross-section or the structure will crack or collapse. This is why brick chimneys and walls sometimes blow over in high winds.

This way of combining stresses makes it easy to check that the stresses in a structure are within the limits of the usable stress for the material. Of course all parts of every load path have to be checked in this way for all load combinations to ensure that the stresses in the structure are always within the limits of the material.