they developed since the late 1930s to teach mathematical methods to physics graduate students, they laid out the four properties that a Green's function must possess. Using the Sturm-Liouville problem given by

$$\frac{d}{dx}\left[f(x)\frac{dy}{dx}\right] + p(x)y = -q(x), \tag{1.1.5}$$

these four properties are:

- The Green's function satisfies the homogeneous differential equation when $x \neq \xi$, the source point.
- The Green's function satisfies homogeneous boundary conditions.
- The Green's function is symmetric in the variables x,ξ .
- The Green's function $g(x|\xi)$ satisfies the condition

$$\left.\frac{dg}{dx}\right|_{x=\xi^+} - \left.\frac{dg}{dx}\right|_{x=\xi^-} = -\frac{1}{f(\xi)}. \tag{1.1.6}$$

Prior to the publication of Morse and Feshbach's notes, authors used various tricks to find Green's functions that satisfied these four properties. Morse and Feshbach's great contribution was to show that the "Green's function is the point source solution [to a boundary-value problem] satisfying appropriate boundary conditions." Thus the Green's function could be found by simply solving (in the case of Sturm-Liouville problem)

$$\frac{d}{dx}\left[f(x)\frac{dg}{dx}\right] + p(x)g = -\delta(x-\xi) \tag{1.1.7}$$

with homogeneous boundary conditions, where $\delta(x - \xi)$ was the recently introduced delta function by Dirac. The advantage of this formulation was that the powerful techniques of eigenvalue expansions and transform methods could be used in a straightforward manner to find Green's functions. They will be the primary methods used in this book.

By the 1960's many textbooks began to champion the use of Green's functions. For example, in Mackie's 1965 book[5] he sought "to give a general account of how certain mathematical techniques, notably those of Green's functions and of integral transforms, can be used to solve important and commonly occurring boundary value problems in ordinary and partial differential equations." In the following sections we turn to the development of Green's functions as they evolved within each general class of differential equations.

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[5] Mackie, A. G., 1965: *Boundary Value Problems.* Oliver & Boyd, 252 pp.

1.2 POTENTIAL EQUATION

Shortly after the publication of Green's monograph on the European continent, the German mathematician and pedagogue Carl Gottfried Neumann (1832–1925) developed the concept of Green's function as it applies to the two-dimensional (in contrast to three-dimensional) potential equation.[6] He defined the two-dimensional Green's function, showed that it possesses the property of reciprocity, and found that it behaves as $\ln(r)$ as $r \to \infty$. Using elliptic coordinates he rederived Poisson's integral formula and developed an eigenfunction expansion for the two-dimensional Green's function. In 1875 Paul Meutzner (1849–1914) extended Neumann's work.[7] In particular, he obtained the Green's function for the region within an ellipse (*Ellipsenfläche*) and a circle (*Ringfläche*). Finally, in his book on the logarithmic potential, A. Harnack[8] (1851–1888) gave the Green's function for a circle and rectangle.

All of these authors used a technique that would become one of the fundamental techniques in constructing a Green's function, namely eigenfunction expansions. The investigator would first find an eigenfunction expansion that satisfied both the homogeneous differential equation and boundary conditions. The geometry of the problem would determine the coordinate system that was used. Then the Fourier coefficients would be chosen so that the Green's function exhibited the proper behavior (such as $1/r$) near the source point. Later on,[9] John Dougall (1867–1960) derived three-dimensional Green's functions in cylindrical and spherical coordinates.

In 1879 Alfred George Greenhill[10] (1847–1927) applied the method of images to construct the Green's function for a rectangular parallelepiped. Because his results are expressed as an infinite summation of theta functions, it was not very useful and has essentially been forgotten.

Hector Munro Macdonald[11] (1865–1935) took a slightly different approach in the 1890s. As before, he began with the eigenfunction expansion

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6 Neumann, C., 1861: Ueber die Integration der partiellen Differentialgleichung: $\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0$. *J. Reine Angew. Math.*, **59**, 335–366.

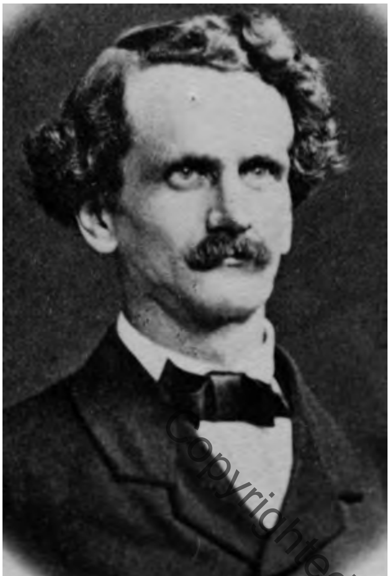
7 Meutzner, P., 1875: Untersuchungen im Gebiete des logarithmischen Potentiales. *Math. Ann.*, **8**, 319–338. For an alternative derivation, see Sections 15 and 17 in Neumann, C., 1906: Über das logarithmische Potential. *Ber. Verh. K. Sachs. Ges. Wiss. Leipzig, Math.-Phys. Klasse*, **58**, 482–559.

8 See Chapter 2 in Harnack, A., 1887: *Die Grundlagen der Theorie des logarithmischen Potentiales und der eindeutigen Potentialfunktion in der Ebene.* Leipzig, B. G. Teubner, 170 pp.

9 Dougall, J., 1900: The determination of Green's function by means of cylindrical or spherical harmonics. *Proc. Edinburgh Math. Soc., Ser. 1*, **18**, 33–83.

10 Greenhill, A. G., 1879: On Green's function for a rectangular parallelepiped. *Proc. Cambridge Philos. Soc.*, **3**, 289–293.

11 Macdonald, H. M., 1895: The electrical distribution on a conductor bounded by two spherical surfaces cutting at any angle. *Proc. London Math. Soc., Ser. 1*, **26**, 156–172;



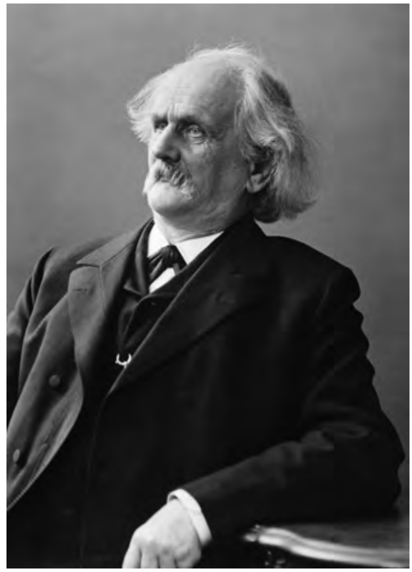


Figure 1.2.1: Carl Gottfried Neumann (1832–1925) was a leading German mathematician and teacher. Today he is best known for his work on the Dirichlet principle and integral equations, and his co-founding with Alfred Clebsch of *Mathematische Annalen*. Left photograph ©Universitätarchiv Leipzig; right photograph ©Photo Deutsches Museum.

that satisfied the boundary conditions. But now, the Fourier coefficients were chosen so that the expansion satisfied the general Poisson equation. Then he considered the special case of a point source. We illustrate his method in Example 6.8.2. Because you must solve the general Poisson equation first, his technique never became popular.

In the late 1890's Arnold Sommerfeld[12] (1868–1951) developed a technique using integration on the complex plane to extend the method of images to several other useful geometries in three dimensions. Ernst William Hobson (1856–1933) then used this method[13] to find the Green's function for a circular disk. Later, Ludwig Waldmann (1913–1980), a young assistant to Sommerfeld, applied this technique in electrostatic calculations of an electron lens.[14] Unfortunately this technique is very complicated and we will present

Macdonald, H. M., 1900: Demonstration of Green's formula for electric density near the vertex of a right cone. *Trans. Cambridge Philos. Soc.*, **18**, 292–297.

[12] Sommerfeld, A., 1897: Über verzweigte Potentiale im Raum. *Proc. London Math. Soc., Ser. 1*, **28**, 395–429.

[13] Hobson, E. W., 1900: On Green's function for a circular disc, with applications to electrostatic problems. *Trans. Cambridge Philos. Soc.*, **18**, 277–291.

[14] Waldmann, L., 1937: Zwei Anwendungen der Sommerfeld'schen Methode der verzweigten Potentiale. *Phys. Z.*, **38**, 654–663.

an improved version in Example 6.2.5.

At the beginning of twentieth century the method of *bilinear expansions* was developed:

$$g(x,y,z|\xi,\eta,\zeta) = \sum_{n=1}^{\infty} \frac{\psi_n(x,y,z)\psi_n(\xi,\eta,\zeta)}{\lambda_n}, \qquad \textbf{(1.2.1)}$$

where λ_n and $\psi_n(x,y,z)$ are the nth eigenvalue and eigenfunction, respectively. Adolf Kneser[15] (1862–1930) showed that the Green's function was the symmetric kernel of the integral equation

$$\psi_n(\xi,\eta,\zeta) = \lambda_n \iiint g(x,y,z|\xi,\eta,\zeta)\psi_n(x,y,z)\,dx\,dy\,dz. \qquad \textbf{(1.2.2)}$$

Assuming that the Green's function can be expressed as an eigenfunction expansion, Equation 1.2.1 follows. As examples, Kneser found the bilinear expansion for rectangular and circular areas and for the surface of a sphere.

In summary then, by 1950 there were essentially three methods[16] for finding Green functions. The first method simply used a Green's function developed for Helmholtz's equation $\nabla^2 u + k_0^2 u = 0$ and took the limit as $k_0 \to 0$. The second method wrote the Green's function as a sum of eigenfunctions that satisfied the boundary conditions. The coefficients were then chosen so that the correct singular behavior occurred at the source point. Finally, the third method wrote the Green's function as the sum of the free-space solution plus a harmonic function.[17] The harmonic solution was chosen so that the Green's function satisfied the boundary conditions.

Later on, Kelvin's classic inversion[18] that maps the interior of a circle or sphere to the exterior and *vice versa* was developed to find the Green's function for Poisson's equation. We will illustrate this technique following Equation 6.3.37 and in Section 6.8.

Finally Green's functions have been used to solve mixed boundary-value problems involving the two-dimensional Poisson's equation. These problems occur when the boundary condition changes along a given boundary from a

[15] Kneser, A., 1911: *Integralgleichungen und ihre Anwendungen in der mathematischen Physik*. Braunschweig, 293 pp.

[16] See, for example, Bouwkamp, C. J., and N. G. de Bruijn, 1947: The electrostatic field of a point charge inside a cylinder, in connection with wave guide theory. *J. Appl. Phys.*, **18**, 562–577. This paper is of particular note because of its use of the modern definition of the delta function. See their Equation 41.

[17] Weber, E., 1939: The electrostatic potential produced by a point charge on the axis of a cylinder. *J. Appl. Phys.*, **10**, 663–666.

[18] Thomson, W., 1845: Extrait d'une lettre de M. William Thomson à M. Liouville. *J. Math. Pures Appl.*, **10**, 364–367; Thomson, W., 1847: Extraits de deux lettres addressées à M. Liouville. *J. Math. Pures Appl.*, **12**, 256–264.

Dirichlet condition to a Neumann condition and *vice versa*. We will explore this topic in Section 6.10.

1.3 HEAT EQUATION

The development of using Green's function to solve the heat equation consists of two parts. In the nineteenth century a synthetical method was developed which replaces the actual distribution by sets of sources and sinks distributed over the boundaries and throughout the region under investigation. The twentieth century has been dominated by the use of transform methods.

Our tale begins with William Thomson (Lord Kelvin)[19] (1824–1907) and his solution of the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < \infty, \quad 0 < t, \tag{1.3.1}$$

with the boundary conditions

$$u(0,t) = \begin{cases} V, & 0 < t < T, \\ 0, & T < t, \end{cases} \qquad \lim_{x\to\infty} u(x,t) \to 0, \qquad 0 < t, \tag{1.3.2}$$

and the initial conditions

$$u(x,0) = 0, \qquad 0 < x < \infty. \tag{1.3.3}$$

Using Fourier integrals, Kelvin obtained the solution

$$u(x,t) = \frac{Vx}{2\sqrt{\pi}} \int_0^T \frac{d\tau}{(t-\tau)^{3/2}} \exp\left[-\frac{x^2}{4(t-\tau)}\right]. \tag{1.3.4}$$

He also showed that the solution could also be obtained "synthetically" for small T if he introduced a source at $x = 0$ and a sink at $x = \alpha$ and took the limit as $\alpha \to 0$. Furthermore, reporting on some correspondence with George G. Stokes (1819–1903), he gave the solution in a form that we now call the superposition integral:

$$u(x,t) = \frac{x}{2\sqrt{\pi}} \int_0^t \exp\left[-\frac{x^2}{4(t-\tau)}\right] \frac{f(\tau)}{t-\tau}\, d\tau, \tag{1.3.5}$$

where $u(0,t) = f(t)$.

In 1887 Ernst William Hobson[20] (1856–1933) generalized Kelvin's synthetical method to two and three dimensions. In general, his effort must

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[19] Thomson, W., 1854/55: On the theory of the electric telegraph. *Proc. R. Soc. London*, **7**, 382–399.

[20] Hobson, E. W., 1887: Synthetical solutions in the conduction of heat. *Proc. London Math. Soc., Ser. 1*, **19**, 279–299.

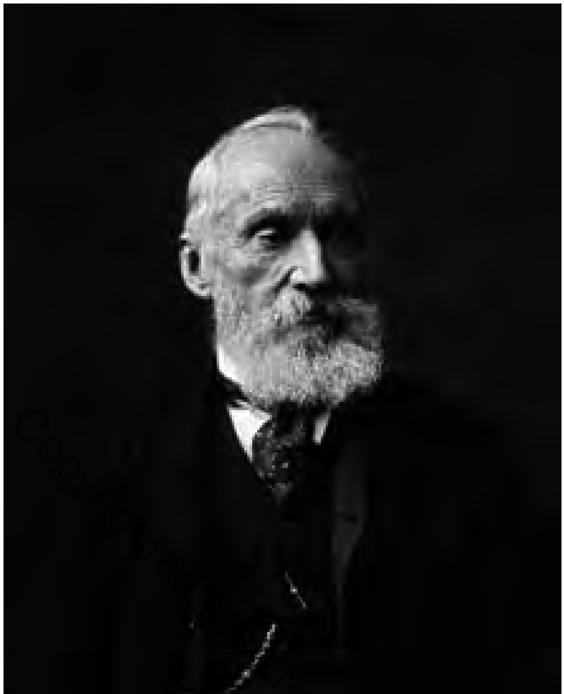

Figure 1.3.1: William Thomson, Baron Kelvin of Largs, (1824–1907) was one of the leading mathematical physicists of the nineteenth century. During his work on thermodynamics he realized that there was a lower limit to temperature. He became well known to the public for his prediction on the speed of a signal in a transatlantic submarine cable that was being laid in the 1850s. (Portrait courtesy of the Royal Society.)

be considered a failure since it resulted in expressions that were difficult to interpret. For example, in another paper,[21] Hobson redid Kelvin's problem of one-dimensional heat conduction when the boundary condition at $x = 0$ changed to $u_x(0,t) = h[u(0,t) - f(t)]$, where h denotes the external conductivity. He found that

$$u(x,t) = \frac{h}{2\sqrt{\pi}} \int_0^\infty \int_0^\infty e^{-h\zeta} \left\{ \exp\left[-\frac{(x+\zeta-\xi)^2}{4t} \right] - \exp\left[-\frac{(x+\zeta+\xi)^2}{4t} \right] \right\} \times [u(\xi,0) - u_x(\xi,0)] \, d\xi d\,\zeta. \quad (\mathbf{1.3.6})$$

Hobson's solution fails if the initial temperature distribution is discontinuous.

In 1892, George H. Bryan[22] (1864–1928) improved the synthetical method. He cleverly wrote the solution as the sum of two parts: a source term

[21] Hobson, E. W., 1888: On a radiation problem. *Math. Proc. Cambridge Philos. Soc.*, **6**, 184–187.

[22] Bryan, G. H., 1892: Note on a problem in the linear conduction of heat. *Math. Proc. Cambridge Philos. Soc.*, **7**, 246–248.

(located at $x = \xi$) plus a homogeneous solution so that the total solution satisfies the boundary condition. Using this technique to redo Hobson's 1888 calculation, he found that

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_0^\infty \left\{ \exp\left[-\frac{(x-\xi)^2}{4t} \right] + \exp\left[-\frac{(x+\xi)^2}{4t} \right] \right\} f(\xi)\, d\xi$$
$$- \frac{h}{\sqrt{\pi t}} \int_0^\infty \int_0^\infty e^{-h\zeta} \exp\left[-\frac{(x+\zeta+\xi)^2}{4t} \right] f(\xi)\, d\xi d\,\zeta. \qquad (\mathbf{1.3.7})$$

Later on, Sommerfeld[23] generalized Kelvin's results by considering substances that have different thermal properties. He considered two cases: (a) two semi-infinite domains with the interface at $x = 0$ and (b) two semi-infinite slabs separated a finite layer lying between $a < x < b$. In 1939 P. V. Solovieff[24] used mirror images to construct Green's functions for a $n+1$ dimensional heat equation when one or more boundaries are moving.

By the turn of the twentieth century, John Dougall[25] (1867–1960) introduced the concept of contour integration to find the mathematical description of how a sphere cools in a well-stirred liquid. Although Dougall's analysis made no use of Green's functions, it provided the necessary insight that allowed H. S. Carslaw to synthesize all of the ideas on how to apply Green's function to heat conduction problems. Carslaw's 1902 paper[26] began by deriving how any conduction problem without sources can be expressed in terms of its Green's function, the initial condition and the solution's value along the boundary surrounding the domain of interest. The question then turned on the question of finding the Green's function for various geometries and boundary conditions. In the case of unbounded domains, he used the Green's functions given by the synthetical method. The remaining Green's functions were obtained using Dougall's method of contour integration. For example, to find the Green's function for linear heat flow over the interval $(0, a)$, when the Green's function vanishes at both ends, Carslaw first introduced the Green's function

$$g_1(x,t|\xi,0) = \frac{1}{2\sqrt{\pi\kappa t}} \left\{ \exp\left[-\frac{(x-\xi)^2}{4\kappa t} \right] - \exp\left[-\frac{(x+\xi)^2}{4\kappa t} \right] \right\}. \qquad (\mathbf{1.3.8})$$

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[23] Sommerfeld, A., 1894: Zur analytischen Theorie der Wärmeleitung. *Math. Ann.*, **45**, 263–277.

[24] Solovieff, P. V., 1939: Die Greensche Funktion der Wärmeleitungsgleichung. *Dokl. Acad. Sci. USSR*, **23**, 132–134; Solovieff, P. V., 1939: Fonctions de Green des équations paraboliques. *Dokl. Acad. Sci. USSR*, **24**, 107–109.

[25] Dougall, J., 1901: Note on the application of complex integration to the equation of conduction of heat, with special reference to Dr. Peddie's problem. *Proc. Edinburgh Math. Soc., Ser. 1*, **19**, 50–56.

[26] Carslaw, H. S., 1902: The use of Green's functions in the mathematical theory of the conduction of heat. *Proc. Edinburgh Math. Soc., Ser. 1*, **21**, 40–64.

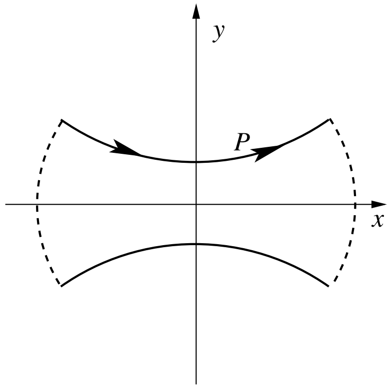

Figure 1.3.2: The contour P and closed contour used in Equation 1.3.9 and Equation 1.3.10, respectively.

The first term is the free-space Green's function given by Kelvin while the second term assures that $g_1(0, t|\xi, 0) = 0$. Carslaw then showed that this Green's function could be expressed by the contour integral

$$g_1(x, t|\xi, 0) = \frac{1}{\pi i} \int_P e^{-\kappa z^2 t} \sin(x_< z) e^{-izx_>} \, dz, \tag{1.3.9}$$

where P is the contour shown in Figure 1.3.2. On the right side, the contour must lie between $0 < \arg(z) < \pi/4$ as $|z| \to \infty$ while on the left side the contour lies between $3\pi/4 < \arg(z) < \pi$ as $|z| \to \infty$.

Next, Carslaw introduced the new Green's function

$$g_2(x, t|\xi, 0) = -\frac{1}{\pi i} \int_P e^{-\kappa z^2 t} \frac{\sin(x_< z) \sin(x_> z)}{\sin(az)} e^{iaz} \, dz. \tag{1.3.10}$$

Why did Carslaw create this new Green's function? Because a linear combination of $g_1(x, t|\xi, 0)$ and $g_2(x, t|\xi, 0)$ yields the Green's function

$$g(x, t|\xi, 0) = \frac{1}{\pi i} \int_P e^{-\kappa z^2 t} \frac{\sin(x_< z) \sin[(a - x_>)z]}{\sin(az)} \, dz \tag{1.3.11}$$

which satisfies the boundary conditions $g(0, t|\xi, 0) = g(a, t|\xi, 0) = 0$. Furthermore, Carslaw was also able to show that this Green's function satisfied the initial condition and had the correct behavior at the singularity $x = \xi$. Using Cauchy's residue theorem and the closed contour showed in Figure 1.3.2, this Green's function could be rewritten in the convenient form of

$$g(x, t|\xi, \tau) = \frac{2}{a} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi\xi}{a}\right) \exp\left[-\frac{\kappa n^2 \pi^2 (t - \tau)}{a^2}\right]. \tag{1.3.12}$$

The difficulty of this method is quite apparent: It is not easy to choose a complex representation for the Green's function that satisfies the boundary conditions, initial condition and the singular nature at the point of excitation. This difficulty became academic with the advent of Laplace transforms.

In the mid-1920s, Gustav Doetsch (1892–1977) wrote a series of papers on heat conduction.[27] In particular, he considered the heat conduction problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < c, \quad 0 < t, \tag{1.3.13}$$

with the boundary conditions

$$u(0,t) = A(t), \quad u(c,t) = B(t), \qquad 0 < t, \tag{1.3.14}$$

and the initial condition

$$u(x,0) = \Phi(x), \qquad 0 < x < c. \tag{1.3.15}$$

He showed that the solution $u(x,t)$ could be expressed by the one-dimensional version of Equation 5.0.11 (see Problem 1 in Chapter 5) and the Green's function

$$g(x,t|\xi,0^+) = \frac{2}{c}\sum_{n=1}^{\infty} e^{-n^2\pi^2 t/c^2} \sin(n\pi\xi/c)\sin(n\pi x/c). \tag{1.3.16}$$

Furthermore, he showed that the Green's function is symmetric. Finally, the limit of $c \to \infty$ yielded the Green's functions on a semi-infinite domain found by Kelvin and Carslaw. The revolutionary aspect of Doetsch's approach was his use of Laplace transforms.

In 1932, Sydney Goldstein[28] (1903–1989) showed how Laplace transforms could be used to solve many heat conduction problems whose derivation up to that time had been very cumbersome. Indeed he pointed out that many of the Green's functions found by Carslaw's contour integral method was in reality the "operational method."

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[27] Bernstein, F., and G. Doetsch, 1925: Probleme aus der Theorie der Wärmeleitung. I. Mitteilung. Eine neue Methode zur Integration partieller Differentialgleichungen. Der lineare Wärmeleiter mit verschwindender Anfangstemperatur. *Math. Z.*, **22**, 285–292; Doetsch, G., 1925: Probleme aus der Theorie der Wärmeleitung. II. Mitteilung. Der lineare Wärmeleiter mit verschwindender Anfangstemperatur. Die allgemeinste Lösung und die Frage der Eindeutigkeit. *Math. Z.*, **22**, 293–306; Doetsch, G., 1925: Probleme aus der Theorie der Wärmeleitung. III. Mitteilung. Der lineare Wärmeleiter mit beliebiger Anfangstemperatur. Die zeitliche Fortsetzung des Wärmezustandes. *Math. Z.*, **25**, 608–626; Bernstein, F., and G. Doetsch, 1927: Probleme aus der Theorie der Wärmeleitung. IV. Mitteilung. Die räumliche Fortsetzung des Temperaturablaufs (Bolometerproblem). *Math. Z.*, **26**, 89–98.

[28] Goldstein, S., 1932: Some two-dimensional diffusion problems with circular symmetry. *Proc. London Math. Soc., Ser. 2*, **34**, 51–88.

Consider, for example, the one-dimensional heat conduction problem

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < \infty, \quad 0 < t. \tag{1.3.17}$$

Taking the Laplace transform of Equation 1.3.17, we have that

$$\frac{d^2U}{dx^2} = q^2 U, \qquad\qquad 0 < x < \infty, \tag{1.3.18}$$

where $s = a^2q^2$. The solution to Equation 1.3.18 is

$$U(x,s) = Ae^{-qx}, \tag{1.3.19}$$

where we have discarded the exponentially growing solution as $x \to \infty$. Now,

$$2\int_0^x U(x,s)\,dx = 2A\left(1 - e^{-qx}\right)/q. \tag{1.3.20}$$

To find the Green's function, we must choose A so that it represents an instantaneous plane source of heat from $x = -\infty$ to $x = \infty$ in the limit of $t \to 0$. In that case, the left side of Equation 1.3.20 equals one as $q \to \infty$, or $2A = q$. Therefore,

$$G(x,s|0,0) = \tfrac{1}{2}qe^{-qx}. \tag{1.3.21}$$

Taking the inverse of $G(x,s|0,0)$, we have that

$$g(x,t|0,0) = \frac{1}{2\sqrt{\pi a^2 t}} \exp\left(-\frac{x^2}{4a^2t}\right), \tag{1.3.22}$$

the same result that Kelvin found. Goldstein used similar methods to find the Green's function for an axisymmetric problem in the plane and outside of a cylinder.

During the 1930s and 1940s several authors found the Green's functions for cylindrical and spherical geometries by using Byran's method of writing the Green's function as a sum of a free-space Green's function plus a homogeneous solution which satisfies the boundary conditions. They obtained the homogeneous solution using Laplace transforms. For example,[29] Lowan redid Bryan's original problem of finding the Green's function in a semi-infinite planar solid that is radiating at the $x = 0$ face. Later, Lowan applied this technique to heat conduction in cylindrical[30] and spherical[31] coordinates.

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[29] Lowan, A. N., 1937: On the operational determination of Green's functions in the theory of heat conduction. *Philos. Mag., Ser. 7*, **24**, 62–70.

[30] Lowan, A. N., 1938: On the operational determination of two dimensional Green's function in the theory of heat conduction. *Bull. Amer. Math. Soc.*, **44**, 125–133.

[31] Lowan, A. N., 1939: On Green's functions in the theory of heat conduction in spherical coordinates. *Bull. Amer. Math. Soc.*, **45**, 310–315 and 951–952.

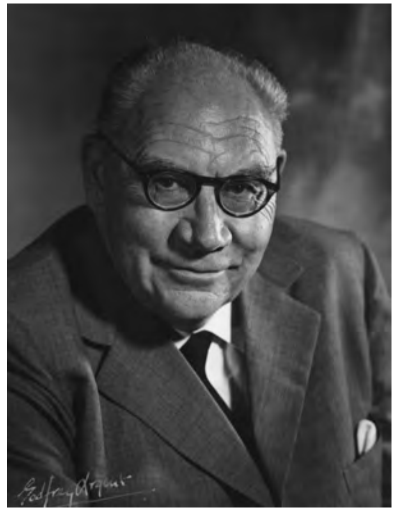

Figure 1.3.3: John Conrad Jaeger's (1907–1979) (right portrait) association with Horatio Scott Carslaw (1870–1954) began during Jaeger's undergraduate education at the University of Sydney. After Jaeger's undergraduate education and subsequent studies at the University of Cambridge, the contact with Carslaw was renewed by irregular trips to Carslaw's retirement farm when Jaeger returned to Tasmania in 1936. These meetings lead to a collaboration on the application of Laplace transforms to find the Green's function for the heat equation. (Carslaw's portrait courtesy of the University of Sydney Archives, Image G3/224/0695; Jaeger's portrait courtesy of the Royal Society.)

During this same period Carslaw and Jaeger also found Green's functions using Laplace transforms. In their earliest paper[32] they found the Green's function for the region outside of a cylinder using both the Laplace transforms and the contour method. In a subsequent paper Carslaw[33] found the Green's function for heat conduction in two semi-infinite solids of different materials with a common boundary at $x = 0$ as well as the case when the two semi-infinite solids are separated by a third solid of thickness $2a$. In the case of three dimension problems, they[34] wrote the Green's function as a sum of the free-space Green's function plus a homogeneous solution of the heat equation. They then used Laplace transforms to find the homogeneous solution. Finally Carslaw and Jaeger[35] applied Bryan's technique to cylindrical problems where

[32] Carslaw, H. S., and J. C. Jaeger, 1939: On Green's functions in the theory of heat conduction. *Bull. Amer. Math. Soc.*, **45**, 407–413.

[33] Carslaw, H. S., 1940: A simple application of the Laplace transformation. *Philos. Mag., Ser. 7*, **30**, 414–417.

[34] Carslaw, H. S., and J. C. Jaeger, 1940: The determination of Green's function for the equation of conduction of heat in cylindrical coordinates by the Laplace transformation. *J. London Math. Soc.*, **15**, 273–281.

[35] Carslaw, H. S., and J. C. Jaeger, 1941: The determination of Green's function for

they used line sources for the free-space Green's functions.

1.4 HELMHOLTZ'S EQUATION

A partial differential equation which is quite similar to Laplace's equation is Helmholtz's equation. It arises in the study of forced (steady-state) vibrations governed by the wave equation; the most famous application is the diffraction of acoustic and visible light waves.

The history of Green's function involving Helmholtz's equation begins with the theoretical work of Hermann von Helmholtz (1821–1894) during his study of acoustics.[36] He showed that the free-space Green's function is $g(x, y, z|\xi, \eta, \zeta) = \cos(k_0 r)/r$, where $r^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2$. Helmholtz used this Green's function to express the solution of $\nabla^2 u + k_0^2 u = 0$ in a region R with the boundary S which gives the solution and its derivative on the boundary.

F. Pockels (1865–1913) epic book[37] on Helmholtz's equation summarized our knowledge of Green's function at the end of the nineteenth century. In Part IV (*Bestimmung der Functionen u aus gegebenen Randwerthen und verwandten Bedingungen*) he reviewed the Dirichlet principle and showed how Green's functions could be used to solve this problem. Next Pockels discussed the expansion of Green's function in term of eigenfunctions. For example, he gave the Green's function within a circle in terms of Fourier series.

For us, Section 4 (*Lösung der Randwerthaufgaben für die Functionen u mit Hülfe verallgemeinerter Green'scher Functionen*) of Part IV is of particular interest. In this section Pockels gave the free-space Green's function in three-dimensional space and on xy-plane, discussed reciprocity and presented the boundary-value solution in terms of the Green's function and the type of boundary condition. In summary, the nineteenth century saw the full development of the concept of the Green's functions but presented few actual functions. This began to change in the twentieth century.

In the early twentieth century several major studies appeared on the Green's functions for Helmholtz's equation. The first paper[38] was by A. Sommerfeld (1868–1951). It consisted of two parts: Green's function for a bounded and unbounded region. For a finite domain he showed that the Green's func-

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line sources for the equation of conduction of heat in cylindrical coordinates by the Laplace transformation. *Philos. Mag., Ser. 7*, **31**, 204–208.

[36] Helmholtz, H., 1860: Theorie der Luftschwingungen in Röhren mit offenen Enden. *J. Reine Angew. Math.*, **57**, 1–72.

[37] Pockels, F., 1891: *Über die partielle Differentialgleichung* $\Delta u + k^2 u = 0$ *und deren Auftreten in der mathematischer Physik*. Leipzig, Teubner, 339 pp.

[38] Sommerfeld, A., 1912: Die Greensche Funktion der Schwingungsgleichung. *Jahresber. Deutsch. Math.-Verein.*, **21**, 309–353.

tion can be expressed by the eigenfunction expansion:

$$g = \sum_m \frac{u_m(O)u_m(P)}{k_0^2 - k_m^2}, \tag{1.4.1}$$

where $u_m(O)$ and $u_m(P)$ are the mth eigenfunction to the eigenvalue problem $\nabla^2 u_m + k_m^2 u_m = 0$ with $u_m = 0$ along the boundary and point O denotes a general point within the domain while the point P is the location of the singularity. Sommerfeld proved that this expansion satisfies the partial differential equation, $g = 0$ at the boundary, is indefinite when point P and point O are collocated, and satisfies reciprocity. As examples, he gave free-space Green's functions in two and three dimensions as well as for a circular membrane with a fixed boundary and a three-dimensional parallelepiped with Neumann boundary conditions. In two subsequent papers H. S. Carslaw[39] extended Sommerfeld's results for a wide variety of three dimensional spaces involving cylindrical and spherical coordinates.

Turning to unlimited domains, Sommerfeld gave the free-space Green's function in two and three dimensions. More importantly, he derived his famous "radiation condition" that required outwardly propagating waves from physical considerations.[40]

By 1950 Green's functions for Helmholtz's equation were used to find the wave motions due to flow over a mountain[41] and in acoustics.[42]

1.5 WAVE EQUATION

Soon after the publication of Green's essay, Green's functions were used to solve the wave equation. In 1860 Bernhard Riemann[43] (1826–1866) applied the method of Green's functions to integrate the hyperbolic equation that describes the propagation of sound waves.[44]

[39] Carslaw, H. S., 1912: Integral equations and the determination of Green's functions in the theory of potential. *Proc. Edinburgh Math. Soc., Ser. 1*, **31**, 71–89; Carslaw, H. S., 1914: The Green's function for the equation $\nabla^2 u + k^2 u = 0$. *Proc. London Math. Soc., Ser. 2*, **15**, 236–257.

[40] See Schot, S. M., 1992: Eighty years of Sommerfeld's radiation condition. *Hist. Math.*, **19**, 385–401.

[41] Lyra, G., 1943: Theorie der stationären Leewellenströmung in freier Atmosphäre. *Zeit. Angew. Math. Mech.*, **23**, 1–28.

[42] Foldy, L. L., and H. Primakoff, 1945: A general theory of passive linear electroacoustic transducers and the electroacoustic reciprocity theorem. I. *J. Acoust. Soc. Am.*, **17**, 109–120.

[43] Riemann, B., 1860: Ueber die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite. *Abh. d. Kön. Ges. der Wiss. zu Göttingen*, **8**, 43–65. An English translation appears in Johnson, J. N., and R. Chéret, 1998: *Classic Papers in Shock Compression Science*. Springer-Verlag, 524 pp.

[44] Mackie, A. G., 1964/65: Green's function and Riemann's method. *Proc. Edinburgh Math. Soc., Ser. 2*, **14**, 293–302.

Figure 1.5.1: Originally drawn to mathematics, Arnold Johannes Wilhelm Sommerfeld (1868–1951) migrated into physics due to Klein's interest in applying the theory of complex variables and other pure mathematics to a range of physical topics from astronomy to dynamics. Later on, Sommerfeld contributed to quantum mechanics and statistical mechanics. (AIP Emilio Segrè Visual Archives, Margrethe Bohr Collection.)

For a linear hyperbolic equation of second order in two independent variables

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + 2a\frac{\partial u}{\partial x} - 2b\frac{\partial u}{\partial y} + cu = 0, \qquad (\mathbf{1.5.1})$$

where x and y are chosen so that the two families of characteristic are $x \pm y =$ constant and a, b and c are functions only of x and y, the solution[45] at the point P with coordinates (ξ,η) is

$$u(\xi,\eta) = \tfrac{1}{2}\left.(uG)\right|_A + \tfrac{1}{2}\left.(uG)\right|_B \qquad (\mathbf{1.5.2})$$
$$+ \tfrac{1}{2}\int_{AB} (Gu_y - uG_y + 2b\,uG)\,dx + (Gu_x - uv_x + 2a\,uG)\,dy.$$

Here G denotes the Riemann-Green function and is given by the adjoint equation

$$\frac{\partial^2 G}{\partial x^2} - \frac{\partial^2 G}{\partial y^2} - 2\frac{\partial(aG)}{\partial x} + 2\frac{\partial(bG)}{\partial y} + cG = 0 \qquad (\mathbf{1.5.3})$$

[45] For the derivation, see Section 73 in Webster, A. G., 1966: *Partial Differential Equations of Mathematical Physics*. Dover, 446 pp.

such that

$$\frac{\partial G}{\partial x} + \frac{\partial G}{\partial y} = (a+b)G \qquad \text{on} \quad y - x = \eta - \xi, \tag{1.5.4}$$

$$\frac{\partial G}{\partial x} - \frac{\partial G}{\partial y} = (a-b)G \qquad \text{on} \quad y + x = \eta + \xi, \tag{1.5.5}$$

and $G(\xi,\eta) = 1$. The values of u and its first derivative are specified along the arc AB which is chosen so that no characteristic cuts it more than at one point; the arcs PA and PB are characteristics. Although there are several methods for finding Riemmann-Green functions,[46] actually finding one is very difficult. The greatest success with this technique involved the equation of telegraphy.[47]

The next important development of Green's functions for the wave equation lies with Gustav Robert Kirchhoff[48] (1824–1887), who used it during his study of the three-dimensional wave equation. Starting with Green's second formula, he was able to show that the three-dimensional Green's function is

$$g(x,y,z,t|\xi,\eta,\zeta,\tau) = \frac{\delta(t-\tau-R/c)}{4\pi R}, \tag{1.5.6}$$

where $R = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}$ (modern terminology). Although he did not call his solution a Green's function,[49] he clearly grasped the concept that this solution involved a function that we now call the Dirac delta function (see pg. 667 of his *Annalen d. Physik*'s paper). He used this solution to derive his famous *Kirchhoff's theorem*, which is the mathematical expression for Huygen's principle: energy always propagates out to infinity.

The early twentieth century saw the development of Laplace transforms to solve the wave equation in addition to the previous known method of Fourier

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[46] See Copson, E. T., 1958: On the Riemann-Green function. *Arch. Rat. Mech. Anal.*, **1**, 324–348.

[47] Picard, É., 1894: Sur l'équation aux dérivées partielles qui se recontre dans la théorie de la propagation de l'électricité. *Acad. Sci., Compt. Rend.*, **118**, 16–17; Bois-Reymond, P. du, 1889: Über lineare partielle Differentialgleichungen zweiter Ordnung. *J. Reine Angew. Math.*, **104**, 241–301; Voigt, W., 1899: Ueber die Aenderung der Schwingungsform des Lichtes beim Fortschreiten in einem dispergirenden oder absorbirenden Mittel. *Ann. Phys.*, **304**, 598–603; Gray, M. C., 1923: The equation of telegraphy. *Proc. Edinburgh Math. Soc., Ser. 2*, **42**, 14–28; Rademacker, H., and R. Iglisch, 1961: Randwertprobleme der partiellen Differentialgleichungen zweiter Ordnung, 779–828 in Frank, Ph., and R. von Mises, 1961: *Die Differential- und Integralgleichungen der Mechanik und Physik. I. Mathematischer Teil*. Dover, 916 pp.; Section 74 in Webster, A. G., 1966: *Partial Differential Equation of Mathematical Physics*. Dover, 446 pp.; Wahlberg, C., 1977: Riemann's function for a Klein-Gordon equation with a non-constant coefficient. *J. Phys., Ser. A*, **10**, 867–878; Asfar, O. R., 1990: Riemann-Green function solution of transient electromagnetic plane waves in lossy media. *IEEE Trans. Electromagn. Compat.*, **EMC-32**, 228–231.

[48] Kirchhoff, G., 1882: Zur Theorie der Lichtstrahlen. *Sitzber. K. Preuss. Akad. Wiss. Berlin*, 641–669; reprinted a year later in *Ann. Phys. Chem.*, *Neue Folge*, **18**, 663–695.

[49] This appears to have been done by Gutzmer, A., 1895: Über den analytischen Ausdruck des Huygens'schen Princips. *J. Reine Angew. Math.*, **114**, 333–337.

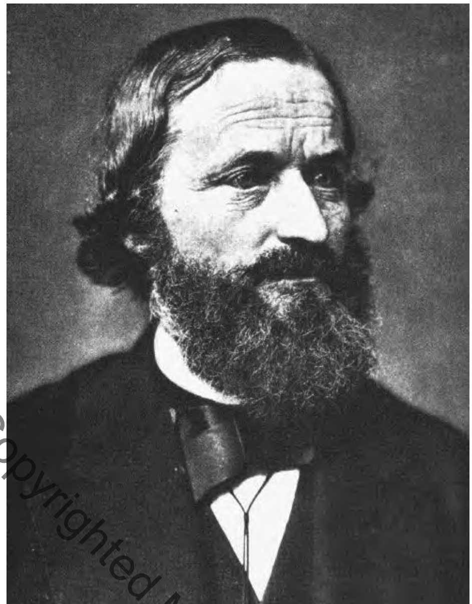

Figure 1.5.2: Gustav Robert Kirchhoff's (1824–1887) most celebrated contributions to physics are the joint founding with Robert Bunsen of the science of spectroscopy, and the discovery of the fundamental law of electromagnetic radiation. Kirchhoff's work on light coincides with his final years as a professor of theoretical physics at Berlin. (Portrait taken from frontispiece of Kirchhoff, G., 1882: *Gesammelte Abhandlungen*. J. A. Barth, 641 pp.)

transforms.[50] In 1914 T. J. I'A. Bromwich (1875–1929) showed how Laplace transforms[51] can be used to solve the wave equation by eliminating the temporal dependence, leaving a boundary-value problem. Interestly he then solved this boundary-value problem using Green's functions. Then, unknowingly he found as an example the Green's function for the one-dimensional wave equation with fixed ends (see his Example 5 on page 438). A. N. Lowan (1898–1962) applied Bromwich's idea to finding the wave motions within a wedge[52] of infinite radius and an infinite solid[53] which is exterior to a cylinder or sphere.

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[50] Poincaré, H., 1893: Sur la propagation de l'électricité. *Acad. Sci., Compt. Rend.*, **117**, 1027–1032; Webster, A. G., 1966: *Partial Differential Equation of Mathematical Physics*. Dover, Section 46.

[51] Bromwich, T. J. I'A., 1914: Normal coordinates in dynamical systems. *Proc. London Math. Soc., Ser. 2*, **15**, 401–448.

[52] Lowan, A. N., 1941: On the problem of wave-motion for the wedge of an angle. *Philos. Mag., Ser.7*, **31**, 373–381.

[53] Lowan, A. N., 1939: On wave motion in an infinite solid bounded internally by a cylinder or a sphere. *Bull. Amer. Math. Soc.*, **45**, 316–325.

One difficulty of finding the Green's function for the wave equation lies in its definition. Around 1950 A. G. Walters wrote a series of papers[54] (that have been essentially forgotten) on finding the Green's function of transient partial differential equations. In the case of the wave equation, his Green's vibrational function $g(P, P_1, t-\tau)$ satisfied the wave equation

$$\frac{\partial^2 g}{\partial t^2} = c^2 D(g), \qquad (\mathbf{1.5.7})$$

where $D(\cdot)$ is a differential operator in one or more dimensions, and satisfies the integral conditions

$$\lim_{\tau\to t} \int_V g(P, P_1, t-\tau)\, dV = 0, \qquad (\mathbf{1.5.8})$$

and

$$\lim_{\tau\to t} \int_V g_t(P, P_1, t-\tau)\, dV = 1, \qquad (\mathbf{1.5.9})$$

where V is any region enclosed by the boundaries and contains the point P_1, which is defined below. He then showed how the solution to the wave equation can be expressed in terms of volume integrals involving the initial conditions, boundary conditions, and any source terms. To compute $g(P, P_1, t-\tau)$, Walters proved that the Green's function is the inverse Laplace transform of $\Phi(P, P_1, s^2/c^2)$, where Φ satisfies the boundary-value problem

$$D(\Phi) = \frac{s^2}{c^2}\Phi. \qquad (\mathbf{1.5.10})$$

Note that Φ satisfies Equation 1.5.10 except for a singular point located at P_1.

Our modern definition of the Green's function for the wave equation as the response of this equation to impulse forcing appears to originate with H. J. Bhabha[55] (1909–1966) in his study of the meson field of neutrons. As we will show in Example 4.1.1, he used transform methods to find the Green's function for the Klein-Gordon equation.

Van der Pol and Bremmer were the first to introduce the general community to the concept that the Green's function[56] of the wave equation is a

[54] Walters, A. G., 1949: The solution of some transient differential equations by means of Green's functions. *Proc. Cambridge Philos. Soc.*, **45**, 69–80; Walters, A. G., 1951: On the propagation of disturbances from moving sources. *Proc. Cambridge Philos. Soc.*, **47**, 109–126.

[55] Bhabha, H. J., 1939: Classical theory of mesons. *Proc. R. Soc. London, Ser. A*, **172**, 384–409.

[56] Van der Pol, B., and H. Bremmer, 1964: *Operational Calculus Based on the Two-Sided Laplace Transform*. Cambridge, 415 pp.

particular solution to the wave equation when it is forced by a point source both in space and time. They then derived the Green's function for the n-dimensional wave equation as well as the three-dimensional wave equation with dispersion. Shortly after Van der Pol's book, P. M. Morse and H. Feshbach further developed the theory of Green's functions as it applies to the wave equation. From Van der Pol's definition, they obtained the reciprocity relation and the free-space Green's function in one, two, and three dimensions.[57] Significantly Morse and Feshbach did not use transform methods but derived their results from heuristic arguments and repeated integration.

The use of transform methods to find Green's function for the wave equation rapidly occurred after the publication of Van der Pol's book. For example, in Friedlander's examination[58] of pulses by a circular cylinder, he found the approximate Green's function for the two-dimensional wave equation exterior to a cylinder of radius 1 where $g_r(1, \theta, t|\rho, \theta', \tau) = 0$. He used many of the techniques in this book. In particular, Laplace transforms were used to eliminate time and Fourier series were employed to give the θ dependence.

An important class of wave propagation involves the diffraction of a direct wave by an infinitesimally thin barrier along the x-axis. In 1935 L. Cagniard used Laplace transforms to find the diffraction of a step function direction by a half-plane.[59] The Green's function follows by simply taking the time derivative of Cagnaird's solution.[60]

The Green's function for the corresponding two-dimensional problem is more difficult. R. D. Turner found the earliest representation using Laplace transforms.[61] G. Schouten[62] has given a closed form solution for the Green's function. See Section 4.8.

1.6 ORDINARY DIFFERENTIAL EQUATIONS

The application of Green's functions to ordinary differential equations began in 1894. Noting the use of Green's functions in solving the two and three

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[57] Morse, P. M., and H. Feshbach, 1953: *Methods of Theoretical Physics. Part I: Chapters 1 to 8.* McGraw-Hill, 997 pp.

[58] Friedlander, F. G., 1954: Diffraction of pulses by a circular cylinder. *Commun. Pure Appl. Math.*, **7**, 705–732.

[59] Cagniard, L., 1935: Diffraction d'une onde progressive par un écran en forme de demi-plan. *J. Phys. Radium, Ser. 7*, **6**, 310–318; Cagniard, L., 1935: Diffraction d'une onde harmonique par un écran en forme de demi-plan. *J. Phys. Radium, Ser. 7*, **6**, 369–372.

[60] Schouten, G., 1999: Two-dimensional effects in the edge sound of vortices and dipoles. *J. Acoust. Soc. Am.*, **106**, 3167–3177.

[61] Turner, R. D., 1956: The diffraction of a cylindrical pulse by a half-plane. *Q. Appl. Math.*, **14**, 63–73.

[62] Schouten, op. cit., p. 3170.

dimensional Poisson equation, H. Burkhardt[63] (1861–1914) asked whether they could be used to solve

$$\frac{d^2y}{dx^2} = f(x), \qquad a < x < b. \tag{1.6.1}$$

He showed that the solution to this problem can be written

$$y(x) = -\int_a^x \frac{(b-x)(\xi-a)}{b-a} f(\xi)\,d\xi - \int_x^b \frac{(b-\xi)(x-a)}{b-a} f(\xi)\,d\xi, \tag{1.6.2}$$

which he wrote

$$y(x) = -\int_a^b g(x|\xi) f(\xi)\,d\xi, \tag{1.6.3}$$

where

$$g(x|\xi) = \frac{(b-x_>)(x_< - a)}{b-a}. \tag{1.6.4}$$

Burkhardt's Green's function $g(x|\xi)$ enjoyed the classic properties:

- The Green's function satisfies the differential equation $g'' = 0$.
- The Green's function is finite and continuous on the interval (a, b) except for $x = \xi$.
- The first derivative of $g(x|\xi)$ is continuous except at $x = \xi$ where it possesses two distinct values that differ by 1.
- $g(a|\xi) = g(b|\xi) = 0$.
- The Green's function has the symmetry property: $g(x|\xi) = g(\xi|x)$.

In 1903 M. Bôcher[64] (1867–1918) gave the properties of the Green's function for homogeneous, linear, n-th order ordinary differential equation

$$\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n y = 0, \qquad a \le x \le b, \tag{1.6.5}$$

where $a_1, a_2, \ldots, a_n$ are real functions of the real variable x and the coefficients of the n linearly independent boundary conditions are constants. William M. Whyburn (1901–1972) would list[65] the properties of the Green's function for the general self-adjoint linear, second-order homogeneous differential equation

$$\frac{d}{dx}\left[p(x,\lambda)\frac{dy}{dx}\right] - q(x,\lambda)y = 0, \qquad a < x < b, \tag{1.6.6}$$

[63] Burkhardt, M. H., 1894: Sur les fonctions de Green relatives a un domaine d'une dimension. *Bull. Soc. Math.*, **22**, 71–75.

[64] Bôcher, M., 1901: Green's function in space of one dimension. *Bull. Amer. Math. Soc., Ser. 2*, **7**, 297–299.

[65] Whyburn, W. M., 1924: An extension of the definition of the Green's function in one dimension. *Ann. Math., Ser. 2*, **26**, 125–130.

Figure 1.6.1: Educated at Harvard and Göttingen, Maxime Bôcher (1867–1918) spent his entire career teaching at Harvard. He published approximately 100 papers on differential equations, series and algebra. Photograph courtesy of the American Mathematical Society (www.ams.org).

with self-adjoint boundary conditions. The jump in the first derivative at the point of excitation $x = \xi$ now becomes $1/p(x,\lambda)$.

Burkhardt also found the Green's function for Equation 1.6.1 when the boundary condition reads

$$y(a) + \chi y'(a) = 0, \qquad y(b) + \chi y'(b) = 0. \tag{1.6.7}$$

In that case,

$$g(x|\xi) = \frac{(b + \chi - x_>)(x_< - a + \chi)}{b - a + 2\chi}. \tag{1.6.8}$$

He noted special difficulties when $\chi = \infty$ and $\chi = (a - b)/2$. In the first case, the boundary condition becomes $g'(a|\xi) = g'(b|\xi) = 0$. For the solution to exist, $\int_a^b f(x)\,dx = 0$. In that case,

$$y(x) = \int_a^x f(\xi)(x - \xi)\,d\xi + \lambda, \tag{1.6.9}$$

where λ is arbitrary. In the second case, the solution exists if

$$\int_a^b \left(x - \frac{a+b}{2}\right) f(x)\,dx = 0. \tag{1.6.10}$$

If this condition holds, then

$$y(x) = \int_a^x f(\xi)(x-\xi)\,d\xi + \lambda\left(x - \frac{a+b}{2}\right). \qquad \textbf{(1.6.11)}$$

All of these results were merely stated, not proved. In 1911 M. Bôcher[66] (1867–1918) provided the mathematical justification for these results.

Ince,[67] in his great treatise on ordinary differential equations, introduced these results to the general community. In his Section 11.1 he proved that the Green's function for Equation 1.6.5 exists and is unique. Furthermore he showed that the Green function is *symmetrical* $g(x|\xi) = g(\xi|x)$ if the ordinary differential equation is self-adjoint. In Section 11.11 Ince further proved that

$$y(x) = \int_a^b g(x|\xi) f(\xi)\,d\xi \qquad \textbf{(1.6.12)}$$

is the solution to the non-homogeneous equation

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n y = f(x), \qquad a \le x \le b. \qquad \textbf{(1.6.13)}$$

As we will see in Section 3.4 an important concept in the theoretical development of Green's functions as they apply to ordinary differential equations involves the *adjoint*. In 1908 G. D. Birkhoff[68] formulated the Green's function in terms of the adjoint differential equations and the eigenvalues and eigenfunctions of the differential equations. He expressed the Green's function as a ratio of determinants in terms of linearly independent solutions to the differential equation and the boundary conditions. Then, in 1909 E. Bounitzky[69] extended Birkhoff results (although he did not reference his paper) to systems of first-order differential equations.

Of all of the possible ordinary differential equations that possess a Green's function, several are of fundamental importance because these arise in important physical problems. One of these is

$$\frac{d^2 u}{dx^2} + [\ell(x) + \lambda^2]u = 0, \qquad 0 < x < 1, \qquad \textbf{(1.6.14)}$$

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[66] Bôcher, M., 1911/12: Boundary problems and Green's functions for linear differential and difference equations. *Ann. Math., Ser. 2*, **13**, 71–88.

[67] Ince, E. L., 1956: *Ordinary Differential Equations*. Dover, 558 pp.

[68] Birkhoff, G. D., 1908: Boundary value and expansion problems of ordinary linear differential equations. *Trans. Amer. Math. Soc.*, **9**, 373–395.

[69] Bounitzky, E., 1909: Sur la fonction de Green des équations différentielles linéaires ordinaires. *J. Math. Pures Appl., Ser. 6*, **5**, 65–126.

with $u(0) = u(1) = 0$. In 1911 Hilb[70] (1882–1929) wrote down the Green's functions in terms of linearly homogeneous solutions to Equation 1.6.14. Furthermore, he found the bilinear representation of the Green's function,

$$g(x|\xi) = \sum_j \frac{\varphi_j(\xi)\varphi_j(x)}{\lambda_j^2 - \lambda^2}, \tag{1.6.15}$$

where λ_j and $\varphi_j(x)$ are the eigenvalues and eigenfunctions, respectively, of the boundary-value problem.

Another differential equation that arises from physics is the simple harmonic oscillator. In this case, the Green's function is governed by

$$g'' + k^2 g = \delta(x - \xi). \tag{1.6.16}$$

In 1910 A. Sommerfeld (1868–1951) examined the solution[71] to Equation 1.6.16 over a finite interval $0 < x < L$ when the boundary conditions are $g(0|\xi) = g(L|\xi) = 0$. He expressed the Green's function by the expansion

$$g(x|\xi) = \sum_{m=1}^{\infty} \frac{u_m(x)u_m(\xi)}{k^2 - k_m^2}, \tag{1.6.17}$$

where k_m and $u_m(x)$ are the eigenvalues and eigenfunctions of the Sturm-Liouville problem: $u_m'' + k^2 u_m = 0$ with $u_m(0) = u_m(L) = 0$.

The bilinear expansion found by Sommerfeld for the harmonic oscillator is a simple example of the general Sturm-Liouville problem:

$$-\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + q(x)u = \lambda u, \qquad 0 < x < 1, \tag{1.6.18}$$

with $u(0) = u(1) = 0$. In 1911, Adolf Kneser[72] (1862–1930) showed how the symmetric Green's function for this problem could be formulated as the homogeneous integral equation

$$u(\xi) = \lambda \int_0^1 g(x|\xi)u(x)\,dx. \tag{1.6.19}$$

He showed that this integral equation lead directly to the general *bilinear formula*

$$g(x|\xi) = \sum_{m=1}^{\infty} \frac{\psi_m(x)\psi_m(\xi)}{\lambda_m}, \tag{1.6.20}$$

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[70] Hilb, E., 1911: Über Reihenentwicklungen nach den Eigenfunktionen linearer Differentialgleichungen 2[ter] Ordnung. *Math. Ann.*, **71**, 76–87.

[71] Sommerfeld, A., 1910: Die Greensche Funktion der Schwingungsgleichungen für ein beliebiges Gebiet. *Phys. Z.*, **11**, 1057–1066.

[72] Kneser, op. cit.

where λ_m and $\psi_m(x)$ are the mth eigenvalue and eigenfunction, respectively, of Equation 1.6.18.

In the following year Sommerfeld considered the case when the interval is infinite $-\infty < x < \infty$. He showed[73] that the Green's function in this case is

$$g(x|\xi) = \frac{1}{2ik} e^{\pm ik(x-\xi)}, \qquad (\mathbf{1.6.21})$$

where the positive sign holds if $x > \xi$ and the negative sign holds when $x < \xi$. To find this solution, Sommerfeld[74] introduced his famous "radiation condition" that requires that energy always propagates out to infinity.

Another important ordinary differential equation whose Green's function was studied in the early twentieth century is the one that governs the deflection of a beam[75] with clamped ends:

$$g^{iv} = -\delta(x-\xi), \qquad 0 < x < L, \qquad (\mathbf{1.6.22})$$

with the boundary conditions

$$g(0|\xi) = g(L|\xi) = g''(0|\xi) = g''(L|\xi) = 0. \qquad (\mathbf{1.6.23})$$

In Chapter 3 we will show that

$$g(x|\xi) = -\frac{x_<(x_> - L)}{6L}\left(x_<^2 + x_>^2 - 2Lx_>\right). \qquad (\mathbf{1.6.24})$$

A quick check shows that Equation 1.6.24 satisfies the ordinary differential equation and boundary conditions. Furthermore, $g'(\xi|\xi)$ and $g''(\xi|\xi)$ is continuous while

$$\left.\frac{d^3g}{dx^3}\right|_{\xi^-}^{\xi^+} = -1. \qquad (\mathbf{1.6.25})$$

In 1914 A. Kneser[76] published a paper on the integral equation that expresses the vibrations of a string where the right boundary is attached to a mass and a spring. His analysis involved solving the Green's function problem (in modern notation) of

$$g'' + \lambda^2 g = -\delta(x-\xi), \qquad 0 < x, \xi < L, \qquad (\mathbf{1.6.26})$$

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[73] Sommerfeld, A., 1912: Die Greensche Funktion der Schwingungsgleichung. *Jahrber. Deutsch. Math.-Verein.*, **21**, 309–353.

[74] Schot, S. M., 1992: Eighty years of Sommerfeld's radiation condition. *Hist. Math.*, **19**, 385–401.

[75] See Section 1.22 and 1.23 in Bateman, H., 1959: *Partial Differential Equations of Mathematical Physics*. Cambridge, 522 pp. See also Von Mises, R., Ph. Frank, H. Weber, and B. Riemann, 1925: *Die Differential- und Integralgleichungen der Mechanik und Physik. Vol I.* Braunschweig, F. Vieweg, 687 pp.

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with

$$g(0|\xi) = g'(L|\xi) + (p - q\lambda^2)g(L|\xi) = 0, \qquad (\mathbf{1.6.27})$$

where $\lambda \neq \lambda_n$, the eigenvalue of the system, and $q > 0$. This problem is of particular interest because λ appears in the differential equation *and* the boundary condition. He found that the corresponding Green's function is

$$g(x|\xi) = \frac{\{(q\lambda^2 - p)\sin[\lambda(L - x_>)] - \lambda\cos[\lambda(L - x_>)]\}\sin(\lambda x_<)}{\lambda(q\lambda^2 - p)\sin(\lambda L) - \lambda^2\cos(\lambda L)}. \qquad (\mathbf{1.6.28})$$

He also found the bilinear expansion for this Green's function.

In the papers cited above, the differential equations were incompatible - there are no nonvanishing solutions which satisfy both the homogeneous differential equation and the boundary conditions. However, as early as 1904, Hilbert[77] gave simple examples of compatible differential equations for linear second-order, boundary-value problems of the form

$$L[u] = [p(t)u']' - q(t)u = 0. \qquad (\mathbf{1.6.29})$$

He then constructed a *generalized Green's function* (*Greensche Funktionen im erweiterten Sinne*) to treat such cases. In 1909, Westfall[78] proved Hilbert's results. Almost two decades later, Elliott[79] generalized Hilbert's results by finding the generalized Green's function for a nth order differential system. His results are not limited to self-adjoint systems.

Since these early studies, the study of generalized Green's functions has languished. Loud[80] used regular differential operator theory to explain generalized Green's functions and developed techniques to find them. In 1977 Locker[81] explored the properties of generalized Green's functions.

[77] See pp. 44−45 in Hilbert, D., 1912: *Grundzüge einer allgemeiner Theorie der linearen Integralgleichungen.* Leipzig, B. G. Teubner, 312 pp.

[78] Westfall, W. D. A., 1909: Existence of the generalized Green's function. *Ann. Math., Ser. 2*, **10**, 177–180.

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[80] Loud, W. S., 1970: Some examples of generalized Green's functions and generalized Green's matrices. *SIAM Rev.*, **12**, 194–210.

[81] Locker, J., 1977: The generalized Green's function for the nth order linear differential operator. *Trans. Amer. Math. Soc.,* **228**, 243–268.